Module 1 Lesson 1
Function and Function Notation

Objectives

In this lesson, we will learn how to

- define a relation and identify its domain and range;
- decide whether a relation is a function;
- use the vertical line test;
- find x- and y-intercepts using function notation;
- evaluate a function at a given input.

1. Relations, Domain, and Range

A relation in x and y is a set of ordered pairs (x, y).

Domain and Range

- The set of x-values in the ordered pairs is called the domain of the relation.
- The set of y-values in the ordered pairs is called the range of the relation.

2. Functions

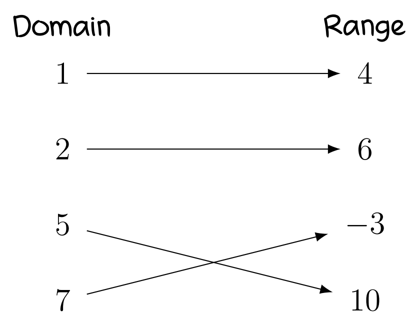
Given a relation in x and y, we say that y is a function of x if for each value of x in the domain, there is exactly one value of y in the range.

Example of a function:

Domain $= \{1, 2, 5, 7\}$ (independent variables, x's),

Range $= \{4, 6, -3, 10\}$ (dependent variables, y's),

paired as shown below:

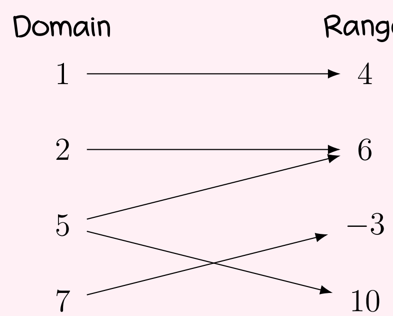


Example 1: Consider the domain and range

Domain $= \{1, 2, 5, 7\}$ (independent variables, x's),

Range $= \{4, 6, -3, 10\}$ (dependent variables, y's),

paired as shown below:



This gives the ordered pairs

$$(1, 4), \quad (2, 6), \quad (5, 6), \quad (7, -3), \quad (5, 10).$$

Is this a function?

Solution

Look at the x-value 5. It is paired with both 6 and 10:

$$(5, 6) \quad \text{and} \quad (5, 10).$$

One x cannot be paired with more than one y.
Therefore, this relation is not a function.
Note: it is perfectly fine for one y-value to be paired with more than one x-value --- notice 6 is paired with both 2 and 5 above, and that alone does not break the function rule.

Example 2: Consider pairing players on a basketball team with their heights:

Player 1 $\longrightarrow$ $6'2''$

Player 2 $\longrightarrow$ $6'5''$

Player 3 $\longrightarrow$ $6'8''$

Player 4 $\longrightarrow$ $6'4''$

If this relation were not a function, what would have to happen?

Solution

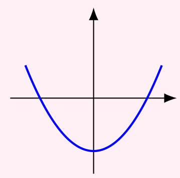
As given, every player is paired with exactly one height, so this relation is a function.
For it to fail to be a function, some player would have to be paired with two different heights --- for example, if Player 1 were listed as both $6'2''$ and $6'5''$.

3. The Vertical Line Test

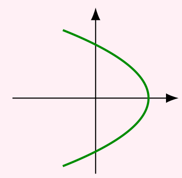
Vertical Line Test

A graph represents a function if and only if no vertical line intersects the graph at more than one point.

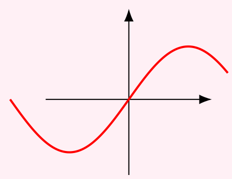
Example 3: Does each graph below represent a function?



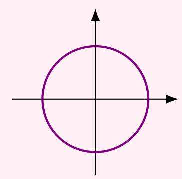
(a)



(b)



(c)



(d)

Solution

(a) Function. Every vertical line crosses the upward-opening parabola exactly once.

(b) Not a function. A vertical line through the middle of the sideways parabola crosses it twice.

(c) Function. Every vertical line crosses the wave exactly once.

(d) Not a function. A vertical line through the interior of the circle crosses it twice.

Example 4: Consider the table

x	-1	6		5
y	12	8	-5	4

If this table were not a function, what would have to happen?

Solution

For the table to fail to represent a function, some x-value would need to appear twice with two different y-values --- for instance, if $x = 6$ were paired with both 8 and -5.

4. Finding x- and y-Intercepts Using Function Notation

Intercepts from $y = f(x)$

- The x-intercepts are the real solutions of $f(x) = 0$. They are the points where the graph crosses the x-axis.
- The y-intercept is given by $f(0)$. It is the point where the graph crosses the y-axis.

Example 5: Find the x- and y-intercepts of

(i) $f(x) = 4x - 12,$ (ii) $f(x) = x^2 - 25.$

Solution

(i) Set $f(x) = 0$:

$$4x - 12 = 0 \implies x = 3.$$

The x-intercept is $(3, 0)$.
The y-intercept is

$$f(0) = 4(0) - 12 = -12,$$

so the y-intercept is $(0, -12)$.
(ii) Set $f(x) = 0$:

$$x^2 - 25 = 0 \implies x^2 = 25 \implies x = \pm 5.$$

The x-intercepts are $(5, 0)$ and $(-5, 0)$.
The y-intercept is

$$f(0) = 0^2 - 25 = -25,$$

so the y-intercept is $(0, -25)$.

5. Function Notation: Naming the Rule

One of the most common ways to represent a function is by a formula or rule. For example,

$$f(x) = x^2 + 1 \qquad \text{or} \qquad y = x^2 + 1.$$

The function's name is f. Its rule says: "to find the number that x is paired with, you square x and then add 1 to the result."

6. Evaluating Functions

Example 6: Evaluate each function at the given input.

(a) If $G(x) = 3x^3 - x - 2$, find $G(2)$.

(b) If $p(r) = \sqrt{r^2 + 7} - 2$, find $p(-3)$.

(c) If $r(s) = \dfrac{4 - s}{2 + s}$, find $r(-2)$.

(d) Let $f(x) = \sqrt{x - 4}$. Determine x for which $f(x) = 3$.

Solution

(a)

$$G(2) = 3(2)^3 - 2 - 2 = 3(8) - 4 = 24 - 4 = 20.$$

(b)

$$p(-3) = \sqrt{(-3)^2 + 7} - 2 = \sqrt{9 + 7} - 2 = \sqrt{16} - 2 = 4 - 2 = 2.$$

(c) Substituting $s = -2$ gives

$$r(-2) = \frac{4 - (-2)}{2 + (-2)} = \frac{6}{0}.$$

Since we cannot divide by zero, $r(-2)$ is undefined. (This is a preview of domain --- see the next lesson!)

(d)

$$\sqrt{x - 4} = 3 \implies x - 4 = 9 \implies x = 13.$$

7. More Practice with Function Notation

Example 7: Given the table representing a function f,

x	-2	-1	3	4	6	10
$f(x)$	4	8	0	-2	8	5

(a) Determine $f(4)$.

(b) Solve $f(x) = 8$ for x.

Solution

(a) Reading the table at $x = 4$:

$$f(4) = -2.$$

(b) We look for x-values whose output is 8. Both $x = -1$ and $x = 6$ give output 8, so

$$x = -1 \qquad \text{or} \qquad x = 6.$$

Example 8: If a rock falls from a height of 100 meters on a planet P, its height H (in meters) after t seconds is approximately

$$H(t) = 100 - 3t^2.$$

(a) What is the height of the rock when $t = 1$, $t = 2$, and $t = 3$?

(b) When is the height of the rock 52 meters?

Solution

(a)

$$H(1) = 100 - 3(1)^2 = 97,$$

$$H(2) = 100 - 3(2)^2 = 100 - 12 = 88,$$

$$H(3) = 100 - 3(3)^2 = 100 - 27 = 73.$$

(b) Set $H(t) = 52$:

$$100 - 3t^2 = 52.$$

$$48 = 3t^2.$$

$$t^2 = 16.$$

$$t = \pm 4.$$

Since time cannot be negative, we reject $t = -4$. The rock is at 52 meters when

$$t = 4 \text{ seconds.}$$

Summary

Key Ideas

- A relation is a set of ordered pairs (x, y); its domain is the set of x-values and its range is the set of y-values.
- A relation is a function if every x-value in the domain corresponds to exactly one y-value. It is fine for a y-value to repeat.
- The vertical line test: a graph is a function if and only if no vertical line crosses it more than once.
- The x-intercepts of $y = f(x)$ solve $f(x) = 0$; the y-intercept is $f(0)$.
- To evaluate a function, substitute the given input for the variable and simplify.