Module 1 Lesson 2
Domain and Range

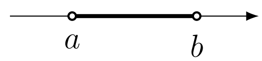
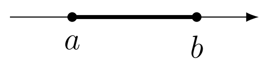
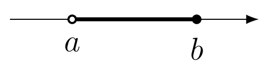
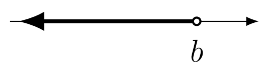
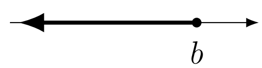
Objectives

In this lesson, we will learn how to

- read and write interval notation;
- determine the domain of a function algebraically;
- determine the domain and range of a function from its graph.

1. Interval Notation

Interval notation is a shorthand for describing a set of real numbers using inequalities.

Inequality	Interval notation	Graph
$a < x < b$	(a, b)	
$a \leq x \leq b$	$[a, b]$	
$a \leq x < b$	$[a, b)$	
$a < x \leq b$	$(a, b]$	
$a < x$	(a, ∞)	
$a \leq x$	$[a, \infty)$	
$x < b$	$(-\infty, b)$	
$x \leq b$	$(-\infty, b]$	

Reading Interval Notation

A parenthesis means the endpoint is not included; a bracket means the endpoint is included. Infinity always gets a parenthesis, since it is not a number that can actually be ``reached.''

2. Domain of a Function

Domain

The domain of a function is the complete set of input values (x-values) for which the function is defined and produces a real number output. Common restrictions include:

(a) avoiding division by zero in rational functions;

(b) avoiding negative values under square roots.

Example 1: Determine the domain of each function.
(Clue: is there any value that x cannot be?)

(i) $f(x) = 3x^2 + 2x - 1$

(ii) $g(x) = \dfrac{5}{x-3}$

(iii) $k(x) = \sqrt{3x - 12}$

(iv) $h(x) = \dfrac{2x}{x^2 - x - 6}$

Challenging one:

$$h(x) = \frac{3}{\sqrt{2x-5}}.$$

Solution

(i) f is a polynomial, so there are no restrictions on x. The domain is

$$(-\infty, \infty).$$

(ii) We need $x - 3 \neq 0$, i.e., $x \neq 3$. The domain is

$$(-\infty, 3) \cup (3, \infty).$$

(iii) We need $3x - 12 \geq 0$, i.e., $x \geq 4$. The domain is

$$[4, \infty).$$

(iv) Factor the denominator:

$$x^2 - x - 6 = (x - 3)(x + 2).$$

We need $x \neq 3$ and $x \neq -2$. The domain is

$$(-\infty, -2) \cup (-2, 3) \cup (3, \infty).$$

Challenging one: Since the square root is in the denominator, we need the radicand to be strictly positive:

$$2x - 5 > 0 \implies x > \frac{5}{2}.$$

The domain is

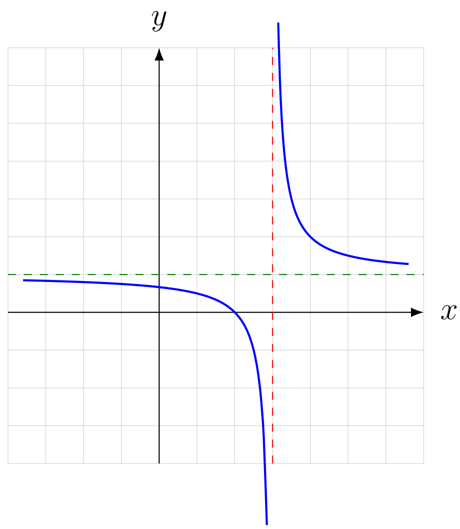
$$\left(\frac{5}{2}, \infty\right).$$

3. Range of a Function

Range

The range is the complete set of output values (y-values) generated by applying the function to the inputs in the domain.

Example 2: Determine the domain and range of the function represented by the graph below.



Solution

The graph never touches the vertical dashed line $x = 3$, so 3 is excluded from the domain:

$$\text{domain} = (-\infty, 3) \cup (3, \infty).$$

The graph gets arbitrarily close to the horizontal dashed line $y = 1$ but never reaches it, so 1 is excluded from the range:

$$\text{range} = (-\infty, 1) \cup (1, \infty).$$

Example 3: Use the pictured function f to find:

(a) $f(2)$

(b) $f(1)$

(c) all x for which $f(x) = 2$

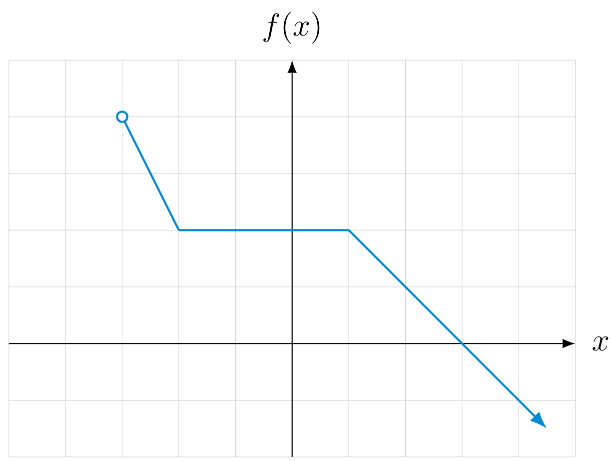
(d) all x for which $f(x) = -1$

(e) the x-intercept(s)

(f) the y-intercept

(g) the domain of f

(h) the range of f



Solution

From $(-3,4)$ (open) to $(-2,2)$, the piece has slope

$$\frac{2-4}{-2-(-3)} = \frac{-2}{1} = -2,$$

so this piece is $f(x) = -2x - 2$.
From $(1,2)$ onward, the piece has slope

$$\frac{0-2}{3-1} = \frac{-2}{2} = -1,$$

so this piece is $f(x) = -x + 3$.
(a) $x = 2$ is on the decreasing piece: $f(2) = -(2) + 3 = 1$.
(b) $x = 1$ is on the flat piece: $f(1) = 2$.
(c) $f(x) = 2$ for every x on the flat segment: $x \in [-2, 1]$.
(d) On the decreasing piece, $-x + 3 = -1$ gives $x = 4$.
(e) The graph crosses the x-axis where $-x + 3 = 0$, i.e., at $(3, 0)$.
(f) $x = 0$ is on the flat piece, so the y-intercept is $(0, 2)$.
(g) The graph starts (excluding) at $x = -3$ and continues without bound to the right:

$$\text{domain} = (-3, \infty).$$

(h) The largest y-value approached (but not attained) is 4; the graph decreases without bound to the right:

$$\text{range} = (-\infty, 4).$$

Summary

Key Ideas

- Interval notation uses parentheses for excluded endpoints and brackets for included endpoints; ∞ always gets a parenthesis.
- The domain of a function excludes values that cause division by zero or a negative number under a square root.
- The range is the set of all output values the function actually produces.
- Domain and range can often be read directly from a graph by looking at its horizontal and vertical extent, and by noting any asymptotes or open circles.