Module 1 Lesson 3
Linear Functions

Objectives

In this lesson, we will learn how to

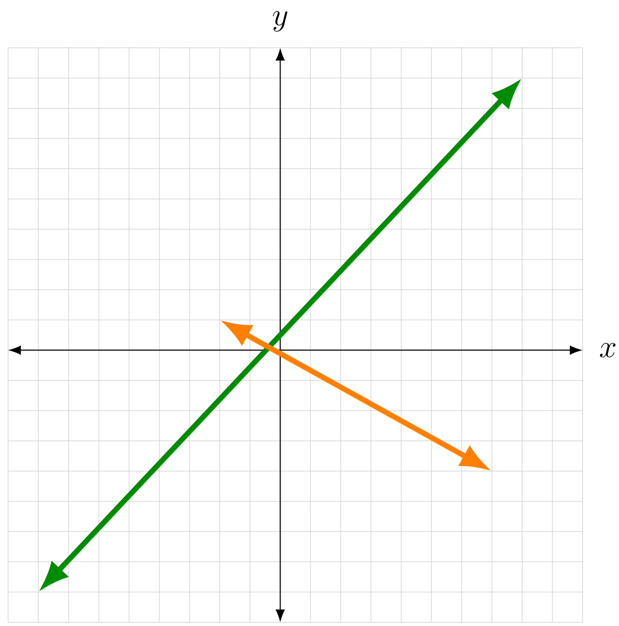
- compute the slope of a line through two points;
- write and graph a line in slope-intercept form;
- compute the average rate of change of a function;
- recognize and apply linear functions to model real situations.

1. Slope of a Line

Slope

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{the change in } y}{\text{the change in } x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

A positive slope rises from left to right; a negative slope falls from left to right.



Example 1: Find the slope of the line through each pair of points.

(1) $(4, 3)$ and $(-2, 15)$

(2) $(6, -3)$ and $(-2, -3)$

(3) $(3, -2)$ and $(3, 6)$

Solution

(1)

$$m = \frac{15-3}{-2-4} = \frac{12}{-6} = -2.$$

(2)

$$m = \frac{-3-(-3)}{-2-6} = \frac{0}{-8} = 0.$$

This is a horizontal line.
(3)

$$m = \frac{6-(-2)}{3-3} = \frac{8}{0},$$

which is undefined. This is a vertical line.

2. Slope-Intercept Form

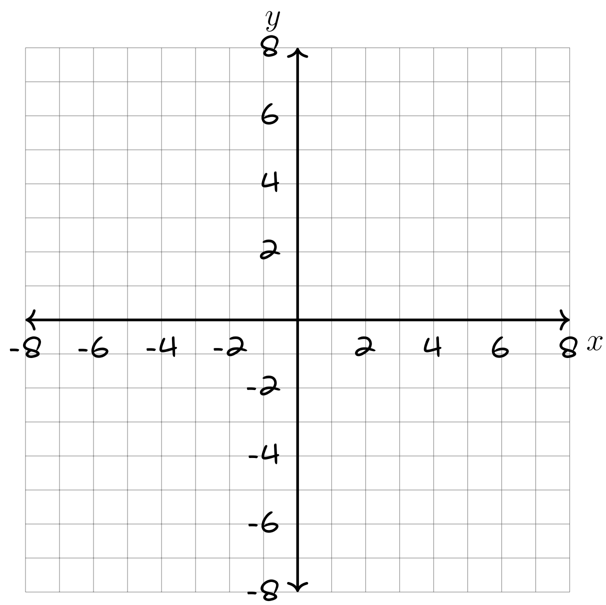
Slope-Intercept Form

A line can be represented by an equation of the form

$$y = mx + b,$$

where m is the slope and b is the y-intercept of the line.

Example 2: Graph $y = \frac{2}{3}x + 4$.



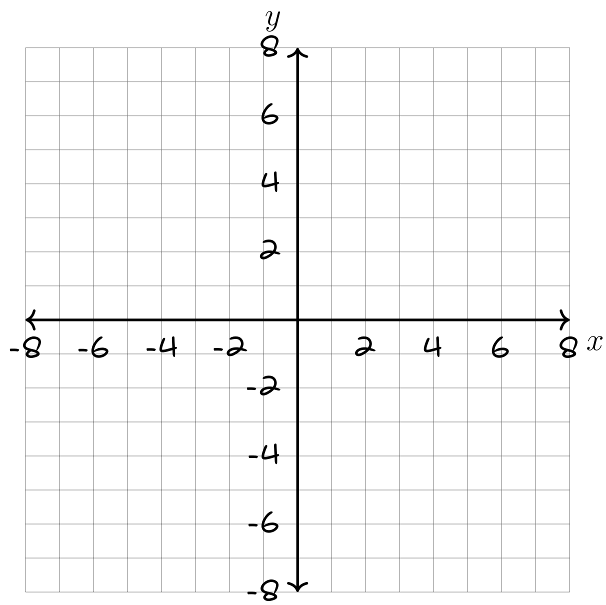
Solution

The y-intercept is $(0, 4)$. From there, use the slope $\frac{2}{3} = \frac{\text{rise}}{\text{run}}$: move up 2 and right 3 to reach a second point, $(3, 6)$. Draw a line through $(0, 4)$ and $(3, 6)$.

Example 3:

(1) Given the equation $3x + 2y = 6$, write the equation in slope-intercept form, determine the slope and y-intercept, and graph the line.

(2) Write an equation of a line in slope-intercept form given a point $(2, -4)$ and slope $m = -3$.


y
8
6
4
2
-8
-6
-4
-2
2
4
6
8
x
-2
-4
-6
-8

Solution

(1) Solve for y:

$$3x + 2y = 6.$$

$$2y = -3x + 6.$$

$$y = -\frac{3}{2}x + 3.$$

The slope is $-\frac{3}{2}$ and the y-intercept is $(0, 3)$. To graph: plot $(0, 3)$, then use the slope $-\frac{3}{2}$ (down 3, right 2, or up 3, left 2) to find a second point and draw the line.

(2) Use point-slope form:

$$y - (-4) = -3(x - 2).$$

$$y + 4 = -3x + 6.$$

$$y = -3x + 2.$$

3. Average Rate of Change

Average Rate of Change (AROC)

The average rate of change of a function over an interval measures the average speed at which the output changes relative to the input.

$$\text{AROC on } [x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Example 4: Given $f(x) = x^2 + 2$, find the average rate of change on $[1, 3]$ and on $[-1, 1]$.

Solution

On $[1, 3]$:

$$\text{AROC} = \frac{f(3) - f(1)}{3 - 1} = \frac{(9+2) - (1+2)}{2} = \frac{11 - 3}{2} = 4.$$

On $[-1, 1]$:

$$\text{AROC} = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{(1+2) - (1+2)}{2} = \frac{0}{2} = 0.$$

4. Linear Functions

Linear Function

A linear function is a function with a constant average rate of change. Its graph is a straight line, written in slope-intercept form as

$$f(x) = mx + b,$$

where m is the rate of change and b is the value of the function when $x = 0$.

Example 5: Determine which of the following relations represents a linear function.

(a)

x	0	1	2	3
$F(x)$	2.5	3	3.5	4

(b)

s	0	2	4	6
$g(s)$	100	90	70	40

(c)

t	10	20	30	40
$h(t)$	5.5	5	4.5	4

Solution

(a) Each step in x increases $F(x)$ by exactly 0.5. The rate of change is constant, so this is linear.
(b) The changes in g are $-10, -20, -30$ for equal steps of 2 in s, giving rates $-5, -10, -15$. The rate is not constant, so this is not linear.
(c) Each step of 10 in t decreases $h(t)$ by exactly 0.5. The rate of change is constant, so this is linear.

5. Applications of Linear Functions

Example 6: A city's population was $30{,}700$ in the year 2000 and is growing by 850 people a year.

(a) Give a formula for the population P as a function of the number of years t since 2000.

(b) What is the population predicted to be in 2012?

(c) When is the population expected to reach $45{,}000$?

Solution

(a)

$$P(t) = 30{,}700 + 850t.$$

(b) 2012 corresponds to $t = 12$:

$$P(12) = 30{,}700 + 850(12) = 30{,}700 + 10{,}200 = 40{,}900.$$

(c) Set $P(t) = 45{,}000$:

$$45{,}000 = 30{,}700 + 850t.$$

$$14{,}300 = 850t.$$

$$t \approx 16.8.$$

The population is expected to reach $45{,}000$ about 16.8 years after 2000, i.e., around the year 2017.

Example 7: Annual revenue R (in billions of dollars) from a restaurant chain worldwide can be estimated by

$$R = 19.1 + 1.8t,$$

where t is the number of years since January 1, 2005.

(a) What is the slope of this function? Interpret it.

(b) What is the vertical intercept? Interpret it.

(c) What annual revenue does the function predict for 2012?

(d) When is annual revenue predicted to hit 35 billion dollars?

Solution

(a) The slope is 1.8 billion dollars per year: revenue increases by about $\$1.8$ billion each year.
(b) The vertical intercept is 19.1 billion dollars (at $t = 0$, i.e., 2005): revenue was about $\$19.1$ billion in 2005.
(c) 2012 corresponds to $t = 7$:

$$R(7) = 19.1 + 1.8(7) = 19.1 + 12.6 = 31.7 \text{ billion dollars.}$$

(d) Set $R = 35$:

$$35 = 19.1 + 1.8t.$$

$$15.9 = 1.8t.$$

$$t \approx 8.8.$$

Revenue is predicted to reach $\$35$ billion about 8.8 years after 2005, i.e., around late 2013.

Example 8: World grain production was 1241 million tons in 1975 and 2048 million tons in 2005, increasing at approximately a constant rate.

(a) Find a linear function $P(t)$ for grain production, in million tons, t years since 1975.

(b) Interpret the slope.

(c) Interpret the vertical intercept.

Solution

(a) The slope is

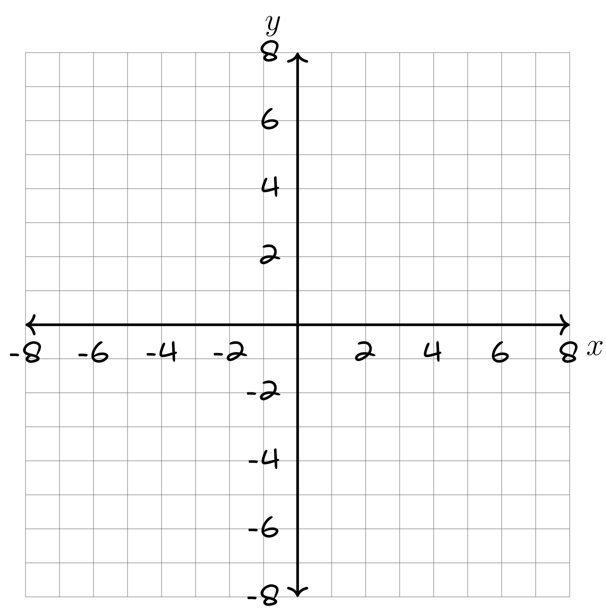
$$m = \frac{2048 - 1241}{2005 - 1975} = \frac{807}{30} = 26.9.$$

Since $P(0) = 1241$,

$$P(t) = 1241 + 26.9t.$$

(b) The slope means grain production increased by about 26.9 million tons each year.
(c) The vertical intercept means grain production was 1241 million tons in 1975.

Example 9: Graph the linear function $f(x) = -\frac{3}{4}x + 4$.



Solution

The y-intercept is $(0, 4)$. Using the slope $-\frac{3}{4}$ (down 3, right 4), a second point is $(4, 1)$. Draw a line through $(0, 4)$ and $(4, 1)$.

Summary

Key Ideas

- Slope: $m = \frac{y_2 - y_1}{x_2 - x_1}$. A horizontal line has slope 0; a vertical line has undefined slope.
- Slope-intercept form: $y = mx + b$.
- Average rate of change on $[x_1, x_2]$ is $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$.
- A linear function has a constant average rate of change, and its graph is a straight line.
- Linear functions are used to model real-world quantities that change at a steady rate.