Module 1 Lesson 6
Absolute Value Equations and Inequalities

Objectives

In this lesson, we will learn how to

- define absolute value as distance from zero;
- solve absolute value equations;
- solve absolute value inequalities;
- apply absolute value to real-world tolerance problems.

1. Absolute Value

Absolute Value

The absolute value of a real number x is its distance from zero, denoted $|x|$. For example,

$$|5| = 5, \qquad |-5| = 5.$$

2. Absolute Value Equations

Solving $|X| = a$

For $a > 0$ and an algebraic expression X,

$$|X| = a \quad \text{is equivalent to} \quad X = -a \quad \text{or} \quad X = a.$$

For example, if $|x| = 5$, then $x = -5$ or $x = 5$.

Example 1: Solve each absolute value equation.

(a) $8|x-3|=4$

(b) $|3x-6|+5=3$

(c) $5-|x+2|=3$

Solution

(a) Divide both sides by 8:

$$|x-3|=\frac{1}{2}.$$

$$x-3=\frac{1}{2} \quad \text{or} \quad x-3=-\frac{1}{2}.$$

$$x=\frac{7}{2} \quad \text{or} \quad x=\frac{5}{2}.$$

(b) Isolate the absolute value:

$$|3x-6|=3-5=-2.$$

An absolute value can never equal a negative number, so there is no solution.

(c) Isolate the absolute value:

$$-|x+2|=3-5=-2 \implies |x+2|=2.$$

$$x+2=2 \quad \text{or} \quad x+2=-2.$$

$$x=0 \quad \text{or} \quad x=-4.$$

3. Absolute Value Inequalities

Solving Absolute Value Inequalities

For $a > 0$ and an algebraic expression X:

$$|X| < a \quad \text{is equivalent to} \quad -a < X < a.$$

$$|X| > a \quad \text{is equivalent to} \quad X < -a \ \text{ or } \ X > a.$$

Example 2: Solve, and write the solution in interval notation.

(a) $|4x| < 12$

(b) $|5x + 1| < 8$

(c) $|5x + 1| > 8$

Solution

(a)

$$-12 < 4x < 12.$$

$$-3 < x < 3.$$

Interval notation: $(-3, 3)$.

(b)

$$-8 < 5x + 1 < 8.$$

$$-9 < 5x < 7.$$

$$-\frac{9}{5} < x < \frac{7}{5}.$$

Interval notation: $\left(-\frac{9}{5}, \frac{7}{5}\right)$.

(c)

$$5x + 1 < -8 \quad \text{or} \quad 5x + 1 > 8.$$

$$5x < -9 \quad \text{or} \quad 5x > 7.$$

$$x < -\frac{9}{5} \quad \text{or} \quad x > \frac{7}{5}.$$

Interval notation: $\left(-\infty, -\frac{9}{5}\right) \cup \left(\frac{7}{5}, \infty\right)$.

4. Applications

Example 3:

(a) Suppose you want your room temperature to stay within 4 degrees of $70°$F. Determine the temperature range using an absolute value inequality.

(b) A machine is designed to produce metal rods that are 10 cm long, with a maximum allowable error of 0.03 cm. Write and solve an absolute value inequality to determine the acceptable rod lengths.

Solution

(a) Let T be the temperature. We need

$$|T - 70| \leq 4.$$

$$-4 \leq T - 70 \leq 4.$$

$$66 \leq T \leq 74.$$

The acceptable temperature range is $66°$F to $74°$F.
(b) Let L be the length of the rod. We need

$$|L - 10| \leq 0.03.$$

$$-0.03 \leq L - 10 \leq 0.03.$$

$$9.97 \leq L \leq 10.03.$$

The acceptable rod lengths are between 9.97 cm and 10.03 cm.

Summary

Key Ideas

- $|x|$ measures the distance of x from 0 on the number line, and is always ≥ 0.
- $|X| = a$ (for $a > 0$) splits into two cases: $X = a$ or $X = -a$.
- If isolating the absolute value produces a negative number on the other side, the equation has no solution.
- $|X| < a$ becomes a ``between'' (double) inequality: $-a < X < a$.
- $|X| > a$ splits into an ``or'' compound inequality: $X < -a$ or $X > a$.
- Absolute value inequalities are natural for describing tolerances and acceptable ranges.