Module 2 Lesson 1
Complex Numbers

Objectives

In this lesson, we will learn how to

- define the imaginary unit i and simplify powers of i;
- write square roots of negative numbers in terms of i;
- add, subtract, multiply, and divide complex numbers;
- find the conjugate of a complex number.

1. The Imaginary Unit

The discussion of complex numbers begins with the imaginary number system. Imaginary numbers received an unfortunate name, as they are often thought of as being ``made-up'' or not real, which is not true.

Today, i is very useful. Engineers use it to study stresses on beams and resonance. Complex numbers help study the flow of fluid around objects, and they are used in electric circuits and in transmitting radio waves. If it weren't for i, we might not be able to talk on cell phones.

The Imaginary Unit i

$$i = \sqrt{-1}.$$

From the definition of i, we notice that

$$i = i,$$

$$i^2 = i \cdot i = \sqrt{-1}\sqrt{-1} = \left(\sqrt{-1}\right)^2 = -1,$$

$$i^3 = i^2 \cdot i = -1 \cdot i = -i,$$

$$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1.$$

Powers of i repeat in a cycle of length 4: $i, -1, -i, 1, i, -1, \ldots$ To simplify a power of i, divide the exponent by 4 and use the remainder.

Example 1: Simplify the following:

(a) i^{22}

(b) i^{68}

Solution

(a) $22 = 4(5) + 2$, so

$$i^{22} = \left(i^4\right)^5 \cdot i^2 = 1^5 \cdot (-1) = -1.$$

(b) $68 = 4(17) + 0$, so

$$i^{68} = \left(i^4\right)^{17} = 1^{17} = 1.$$

2. Square Roots of Negative Numbers

Example 2: Write each expression in terms of i and simplify.

(a) $\sqrt{-16}$

(b) $\sqrt{-28}$

(c) $\sqrt{-50}$

Solution

(a)

$$\sqrt{-16} = \sqrt{16}\sqrt{-1} = 4i.$$

(b)

$$\sqrt{-28} = \sqrt{4 \cdot 7}\sqrt{-1} = 2\sqrt{7}\,i.$$

(c)

$$\sqrt{-50} = \sqrt{25 \cdot 2}\sqrt{-1} = 5\sqrt{2}\,i.$$

A Word of Caution

Recall the multiplication and quotient properties of radicals,

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}, \qquad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}},$$

provided that the roots represent real numbers. If $\sqrt{a}$ and $\sqrt{b}$ are not real numbers, these properties cannot be used --- always convert to i-form first.

Example 3: Multiply or divide the following as indicated:

(a) $\sqrt{-16} \cdot \sqrt{-4}$

(b) $\dfrac{\sqrt{-80}}{\sqrt{-5}}$

Solution

(a) Convert to i-form first:

$$\sqrt{-16} \cdot \sqrt{-4} = (4i)(2i) = 8i^2 = 8(-1) = -8.$$

(b) Convert to i-form first:

$$\frac{\sqrt{-80}}{\sqrt{-5}} = \frac{4\sqrt{5}\,i}{\sqrt{5}\,i} = 4.$$

3. Complex Numbers in Standard Form

Standard Form

The standard form of a complex number is

$$a + bi,$$

where a is the real part and b is the imaginary part. Examples:

$$4 + 3i, \qquad -2 + 6i, \qquad \frac{2}{3} - \frac{4}{5}i.$$

Example 4: Identify the real part and the imaginary part of

$$2 + 5i.$$

If $b = 0$, the number is just a __________ number. If $a = 0$, it is a _______________ number.

Solution

For $2 + 5i$:

$$\text{Real part} = 2, \qquad \text{Imaginary part} = 5.$$

If $b = 0$, the number is just a real number, for example $4 + 0i = 4$.
If $a = 0$, it is a pure imaginary number, for example $0 + 5i = 5i$.

4. Adding and Subtracting Complex Numbers

Adding/Subtracting Complex Numbers

To add or subtract complex numbers, add or subtract the real parts and the imaginary parts separately. Always write the answer in standard form $a \pm bi$.

Example 5: Add the following complex numbers:

(a) $(4 + 30i) + (-3 + 7i)$

(b) $(-5 + 6i) + (12 - 11i)$

Solution

(a)

$$(4 + 30i) + (-3 + 7i) = (4 - 3) + (30 + 7)i = 1 + 37i.$$

(b)

$$(-5 + 6i) + (12 - 11i) = (-5 + 12) + (6 - 11)i = 7 - 5i.$$

5. Multiplying Complex Numbers

Multiplying Complex Numbers

To multiply complex numbers, use the FOIL technique, and use the fact that $i^2 = -1$ to simplify. Always write the final answer in standard form.

Example 6: Multiply: $(-2+6i)(4+3i)$.

Solution

Using FOIL:

$$(-2+6i)(4+3i) = -8 - 6i + 24i + 18i^2.$$

Since $i^2 = -1$:

$$= -8 + 18i + 18(-1) = -8 + 18i - 18 = -26 + 18i.$$

6. Conjugates

Conjugates

The complex numbers $a+bi$ and $a-bi$ are called conjugates of each other: the real parts are the same, but the imaginary parts are of equal magnitude and opposite signs. Multiplying a complex number by its conjugate always produces a real number.

Example 7: Find the conjugate of each complex number.

(a) $2 - 5i$

(b) $-5 + 7i$

(c) $-9i$

Then multiply: $(3 - 4i)(3 + 4i)$.

Solution

(a) Conjugate of $2 - 5i$ is $2 + 5i$.
(b) Conjugate of $-5 + 7i$ is $-5 - 7i$.
(c) Since $-9i = 0 - 9i$, its conjugate is $0 + 9i = 9i$.
Multiply:

$$(3 - 4i)(3 + 4i) = 9 + 12i - 12i - 16i^2 = 9 - 16(-1) = 9 + 16 = 25.$$

Notice the result, 25, is a real number, as expected.

7. Dividing Complex Numbers

Dividing Complex Numbers

To divide by a non-zero complex number, multiply the numerator and denominator by the conjugate of the denominator, then simplify and write the answer in standard form.

Example 8: Divide. Simplify and write the answer in standard form.

(a) $\dfrac{2 + 3i}{3 - 4i}$

(b) $\dfrac{4 - 7i}{5 + 3i}$

Solution

(a) Multiply by the conjugate of the denominator:

$$\frac{2+3i}{3-4i} \cdot \frac{3+4i}{3+4i} = \frac{(2+3i)(3+4i)}{(3-4i)(3+4i)}.$$

Numerator: $6 + 8i + 9i + 12i^2 = 6 + 17i - 12 = -6 + 17i$.
Denominator: $9 - 16i^2 = 9 + 16 = 25$.

$$\frac{2+3i}{3-4i} = \frac{-6+17i}{25} = -\frac{6}{25} + \frac{17}{25}i.$$

(b) Multiply by the conjugate of the denominator:

$$\frac{4-7i}{5+3i} \cdot \frac{5-3i}{5-3i} = \frac{(4-7i)(5-3i)}{(5+3i)(5-3i)}.$$

Numerator: $20 - 12i - 35i + 21i^2 = 20 - 47i - 21 = -1 - 47i$.
Denominator: $25 - 9i^2 = 25 + 9 = 34$.

$$\frac{4-7i}{5+3i} = \frac{-1-47i}{34} = -\frac{1}{34} - \frac{47}{34}i.$$

Summary

Key Ideas

- $i = \sqrt{-1}$, and $i^2 = -1$. Powers of i cycle with period 4: $i, -1, -i, 1$.
- Write $\sqrt{-n}$ (for $n > 0$) as $\sqrt{n}\,i$ before applying radical rules --- the multiplication/quotient properties of radicals only apply to real square roots.
- Standard form of a complex number is $a + bi$, with real part a and imaginary part b.
- Add/subtract complex numbers by combining real and imaginary parts separately.
- Multiply complex numbers with FOIL, using $i^2 = -1$.
- The conjugate of $a + bi$ is $a - bi$; their product is always the real number $a^2 + b^2$.
- Divide complex numbers by multiplying numerator and denominator by the conjugate of the denominator.