Module 2 Lesson 2
Quadratic Equations

Objectives

In this lesson, we will learn how to

- solve quadratic equations using the zero product property;
- solve quadratic equations using the square root property;
- solve quadratic equations by completing the square;
- solve quadratic equations using the quadratic formula;
- use the discriminant to determine the number and type of solutions.

A quadratic equation in the variable x is an equation of the form

$$ax^2 + bx + c = 0,$$

where a, b, and c represent real numbers and $a \neq 0$.

In this lesson, we study several ways to solve quadratic equations, which is also referred to as finding the zeros. The solutions, or zeros, of a quadratic equation represent the points where the graph of the corresponding quadratic function crosses the x-axis.

1. Zero Product Property

Zero Product Property

If $mn = 0$, then $m = 0$ or $n = 0$.

For example,

$$x^2 - 5x + 6 = 0$$

factors as

$$(x-2)(x-3)=0,$$

so $x=2$ or $x=3$.

Example 1: Use the zero product property to solve the following quadratic equations:

(a) $x^2-2x-15=0$

(b) $3x^2+7x-6=0$

Solution

(a) Factor:

$$x^2-2x-15=(x-5)(x+3)=0.$$

So $x-5=0$ or $x+3=0$, giving

$$x=5 \qquad \text{or} \qquad x=-3.$$

(b) We need two numbers whose product is $3\cdot(-6)=-18$ and whose sum is 7: these are 9 and -2.

$$3x^2+9x-2x-6=0.$$

$$3x(x+3)-2(x+3)=0.$$

$$(3x-2)(x+3)=0.$$

So $3x-2=0$ or $x+3=0$, giving

$$x=\frac{2}{3} \qquad \text{or} \qquad x=-3.$$

2. Square Root Property

Square Root Property

If $x^2 = k$, then $x = \pm\sqrt{k}$.

For example,

$$x^2 = 25 \implies x = \pm 5, \qquad (x+5)^2 = 8 \implies x + 5 = \pm\sqrt{8}.$$

3. Completing the Square

A perfect square trinomial can be written as the square of a binomial:

$$x^2 + 10x + 25 \longrightarrow (x+5)^2, \qquad y^2 - 8y + 16 \longrightarrow (y-4)^2.$$

Example 2: Solve the following quadratic equations by completing the square.

(a) $x^2 - 6x + 7 = 0$

(b) $3x^2 + 7x - 8 = 0$

Solution

(a)

$$x^2 - 6x = -7.$$

Add $\left(\frac{-6}{2}\right)^2 = 9$ to both sides:

$$x^2 - 6x + 9 = -7 + 9 = 2.$$

$$(x-3)^2 = 2.$$

$$x - 3 = \pm\sqrt{2}.$$

$$x = 3 \pm \sqrt{2}.$$

(b) Divide by 3:

$$x^2 + \frac{7}{3}x - \frac{8}{3} = 0.$$

$$x^2 + \frac{7}{3}x = \frac{8}{3}.$$

Add $\left(\frac{7}{6}\right)^2 = \frac{49}{36}$ to both sides:

$$x^2 + \frac{7}{3}x + \frac{49}{36} = \frac{8}{3} + \frac{49}{36} = \frac{96}{36} + \frac{49}{36} = \frac{145}{36}.$$

$$\left(x + \frac{7}{6}\right)^2 = \frac{145}{36}.$$

$$x + \frac{7}{6} = \pm\frac{\sqrt{145}}{6}.$$

$$x = \frac{-7 \pm \sqrt{145}}{6}.$$

4. Quadratic Formula

Quadratic Formula

For the quadratic equation written in standard form $ax^2 + bx + c = 0$, the solutions are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 3: Solve the following quadratic equations by using the quadratic formula.

(a) $3x^2 + 7x - 8 = 0$

(b) $5x^2 - 6x = -2$

Solution

(a) Here $a = 3$, $b = 7$, $c = -8$:

$$x = \frac{-7 \pm \sqrt{7^2 - 4(3)(-8)}}{2(3)} = \frac{-7 \pm \sqrt{49 + 96}}{6} = \frac{-7 \pm \sqrt{145}}{6}.$$

This matches the completing-the-square result above --- a good consistency check!

(b) Rewrite in standard form: $5x^2 - 6x + 2 = 0$, so $a = 5$, $b = -6$, $c = 2$:

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(5)(2)}}{2(5)} = \frac{6 \pm \sqrt{36 - 40}}{10} = \frac{6 \pm \sqrt{-4}}{10} = \frac{6 \pm 2i}{10}.$$

$$x = \frac{3}{5} \pm \frac{1}{5}i.$$

5. The Discriminant

Discriminant and Properties of Solutions

For $ax^2 + bx + c = 0$, where a, b, c are real numbers, the discriminant is $b^2 - 4ac$.

(i) $b^2 - 4ac = 0$: one real-number solution;

(ii) $b^2 - 4ac > 0$: two distinct real-number solutions;

(iii) $b^2 - 4ac < 0$: two distinct imaginary solutions, complex conjugates.

For case (ii), if $b^2 - 4ac$ is a perfect square, the solutions are rational; otherwise, they are irrational.

Example 4: Determine the properties of the solutions by computing the discriminant of each quadratic equation.

(a) $4x^2 - 10x + 25 = 0$

(b) $2x^2 - 5x - 3 = 0$

(c) $3x^2 - 2x = -5$

Solution

(a) $a = 4$, $b = -10$, $c = 25$:

$$b^2 - 4ac = 100 - 4(4)(25) = 100 - 400 = -300 < 0.$$

Two distinct imaginary solutions (complex conjugates).

(b) $a = 2$, $b = -5$, $c = -3$:

$$b^2 - 4ac = 25 - 4(2)(-3) = 25 + 24 = 49 > 0,$$

and 49 is a perfect square. Two distinct rational real-number solutions.

(c) Rewrite in standard form: $3x^2 - 2x + 5 = 0$, so $a = 3$, $b = -2$, $c = 5$:

$$b^2 - 4ac = 4 - 4(3)(5) = 4 - 60 = -56 < 0.$$

Two distinct imaginary solutions (complex conjugates).

Summary

Key Ideas

- A quadratic equation has the form $ax^2 + bx + c = 0$, with $a \neq 0$.
- Zero product property: if $mn = 0$, then $m = 0$ or $n = 0$ --- useful when the equation factors nicely.
- Square root property: if $x^2 = k$, then $x = \pm\sqrt{k}$.
- Completing the square turns $x^2 + bx$ into a perfect square trinomial by adding $\left(\frac{b}{2}\right)^2$.
- The quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ solves any quadratic equation.
- The discriminant $b^2 - 4ac$ tells you the number and type of solutions before you even solve.