# Module 2 Lesson 3
# Quadratic Functions and Applications

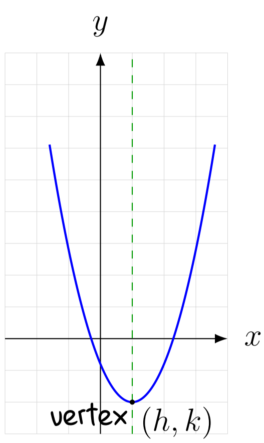
## Objectives

In this lesson, we will learn how to

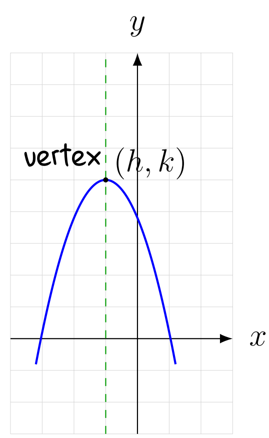
- identify the vertex and axis of symmetry of a parabola;
- find the vertex using a formula and using vertex form;
- sketch a parabola from the sign of $a$, $h$, and $k$;
- solve maximum/minimum application problems.

## 1. Parabolas

The graph of a quadratic function, $f(x) = ax^2 + bx + c$, is a U-shaped curve called a parabola.



opens up: $a > 0$



opens down: $a < 0$

Vertex and Axis of Symmetry

- Vertex : the turning point of the parabola.
- If $a > 0$, the parabola opens upward and the vertex is the minimum point.
- If $a < 0$, the parabola opens downward and the vertex is the maximum point.
- Axis of symmetry : the vertical line through the vertex; the graph is a mirror image across this line.

## 2. Finding the Vertex by Formula

Vertex Formula

For $f(x) = ax^2 + bx + c$, the vertex $(h, k)$ is given by

$$h = \frac{-b}{2a}, \qquad k = f(h).$$

The axis of symmetry is $x = h$.

Example 1: Find the vertex and axis of symmetry of the functions

(a) $f(x) = 3x^2 + 6x + 5$

(b) $f(x) = -2x^2 + 12x + 15$

**Solution**

(a) $a = 3$, $b = 6$:

$$h = \frac{-6}{2(3)} = -1, \qquad k = f(-1) = 3(1) - 6 + 5 = 2.$$

Vertex: $(-1, 2)$. Axis of symmetry: $x = -1$.
(b) $a = -2$, $b = 12$:

$$h = \frac{-12}{2(-2)} = 3, \qquad k = f(3) = -2(9) + 36 + 15 = 33.$$

Vertex: $(3, 33)$. Axis of symmetry: $x = 3$.

## 3. Vertex Form

**Vertex Form**

$$f(x) = a(x - h)^2 + k.$$

- The vertex is $(h, k)$.
- The axis of symmetry is $x = h$.
- If $a > 0$ the graph opens up; if $a < 0$ the graph opens down.

Example 2: Find the vertex, axis of symmetry, and direction of the parabola formed by each function.

(a) $f(x) = -4(x - 3)^2 + 5$

(b) $f(x) = x^2 + 10x + 22$

Solution

(a) Already in vertex form with $a = -4$, $h = 3$, $k = 5$. Vertex: $(3, 5)$. Axis: $x = 3$. Since $a = -4 < 0$, the parabola opens down.
(b) Complete the square:

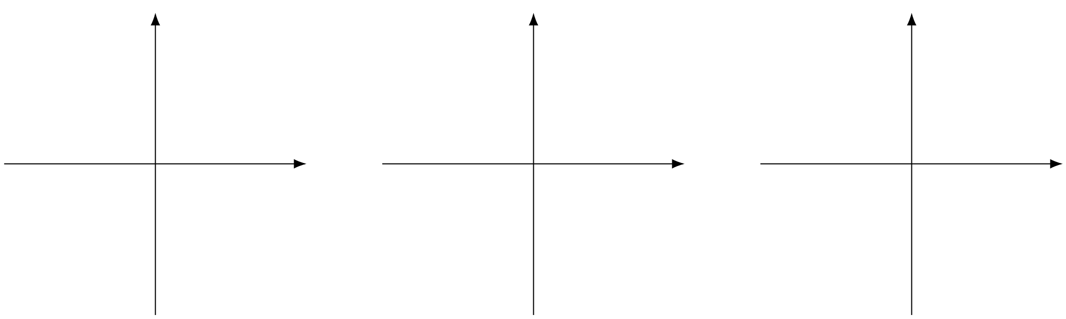
$$x^2 + 10x + 22 = (x^2 + 10x + 25) - 25 + 22 = (x + 5)^2 - 3.$$

So $h = -5$, $k = -3$. Vertex: $(-5, -3)$. Axis: $x = -5$. Since $a = 1 > 0$, the parabola opens up.

## 4. Sketching a Parabola from $a$, $h$, $k$

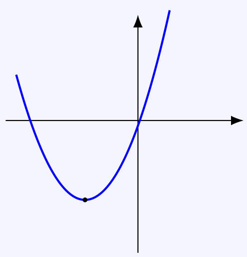
Example 3: Given a quadratic function defined by $f(x) = a(x - h)^2 + k$, determine what the graph would look like based on the given criteria (sketch on the blank axes provided).

1. $a > 0$, $h < 0$, $k < 0$
2. $a < 0$, $h > 0$, $k < 0$
3. $a < 0$, $h < 0$, $k > 0$

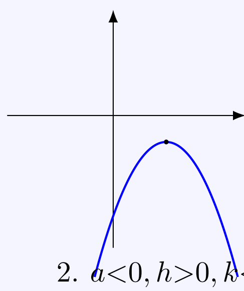


**Solution**

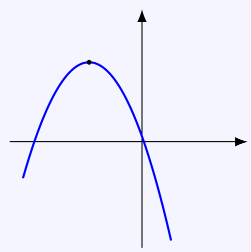
1. $a > 0$ (opens up), vertex $(h, k)$ with $h < 0$, $k < 0$ (vertex in Quadrant III). Since the vertex is below the $x$-axis and the parabola opens up, it crosses the $x$-axis twice.
2. $a < 0$ (opens down), vertex with $h > 0$, $k < 0$ (vertex in Quadrant IV). Since the vertex is already below the $x$-axis and the parabola opens further down, it never crosses the $x$-axis.
3. $a < 0$ (opens down), vertex with $h < 0$, $k > 0$ (vertex in Quadrant II). Since the vertex is above the $x$-axis and the parabola opens down, it crosses the $x$-axis twice.



1. $a>0, h<0, k<0$



2. $a<0, h>0, k<0$



3. $a<0, h<0, k>0$

## 5. Applications: Maximum and Minimum

**Maximum or Minimum Value**

- If the graph opens up, the $k$-value of the vertex is a minimum value.
- If the graph opens down, the $k$-value of the vertex is a maximum value.

**Example 4:** Height of a Missile. A missile is launched with an initial velocity of $150$ ft/sec from a height of $40$ ft. The function

$$H(t) = -5t^2 + 150t + 40$$

gives the height of the missile $t$ seconds after launch. Determine the time at which the missile reaches its maximum height, and find that maximum height.

**Solution**

Here $a = -5$, $b = 150$:

$$h = \frac{-150}{2(-5)} = 15.$$

$$H(15) = -5(15)^2 + 150(15) + 40 = -1125 + 2250 + 40 = 1165.$$

The missile reaches its maximum height of $1165$ feet at $t = 15$ seconds.

**Example 5:** Minimizing Cost. A company's total cost $C(q)$ in dollars for producing $q$ units is modeled by

$$C(q) = 2q^2 - 160q + 5000.$$

Determine the production level at which the cost is minimum, and compute the minimum cost.

**Solution**

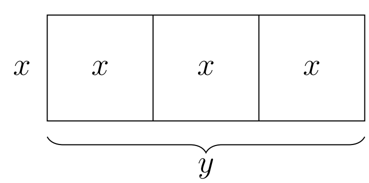
Here $a = 2$, $b = -160$:

$$h = \frac{-(-160)}{2(2)} = \frac{160}{4} = 40.$$

$$C(40) = 2(40)^2 - 160(40) + 5000 = 3200 - 6400 + 5000 = 1800.$$

The minimum cost is $\$1800$, achieved at a production level of $40$ units.

**Example 6:** Maximizing Area. A farmer has $200$ ft of fencing and wants to build three identical adjacent rectangular corrals, as pictured (each corral has depth $x$; the three together span a total width $y$). Determine the dimensions that maximize the total area, and find the area of each individual corral.



Solution

There are $4$ vertical fence segments (each of length $x$) and $2$ horizontal segments (top and bottom, each of length $y$):

$$4x + 2y = 200 \implies y = 100 - 2x.$$

The total area is

$$A(x) = xy = x(100 - 2x) = 100x - 2x^2.$$

This is a downward parabola in $x$, with $a = -2$, $b = 100$:

$$x = \frac{-100}{2(-2)} = 25.$$

Then $y = 100 - 2(25) = 50$.
Maximum total area:

$$A(25) = 25(50) = 1250 \text{ sq ft.}$$

Each individual corral has area

$$\frac{1250}{3} \approx 416.7 \text{ sq ft.}$$

# Summary

**Key Ideas**

- The graph of $f(x) = ax^2 + bx + c$ is a parabola with vertex $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.
- In vertex form $f(x) = a(x-h)^2 + k$, the vertex is $(h, k)$ directly.
- If $a > 0$ the parabola opens up (vertex is a minimum); if $a < 0$ it opens down (vertex is a maximum).
- Many real-world optimization problems (maximizing area, minimizing cost, finding maximum height) reduce to finding the vertex of a quadratic function.