Module 2 Lesson 4
Algebra of Functions and Function Composition

Objectives

In this lesson, we will learn how to

- find the sum, difference, product, and quotient of two functions;
- determine the domain of combined functions;
- find and evaluate the composition of two functions;
- distinguish $(f \circ g)(x)$ from $(fg)(x)$.

1. Sum, Difference, Product, and Quotient of Functions

Algebra of Functions

If $f(x)$ and $g(x)$ are functions, then for all x common to the domains of both functions:

1. Sum: $(f+g)(x) = f(x) + g(x)$
2. Difference: $(f-g)(x) = f(x) - g(x)$
3. Product: $(fg)(x) = f(x) \cdot g(x)$
4. Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$.

Example 1: If $f(x) = x^2 + 1$ and $g(x) = 3x - 5$, find the following and give the domain in interval notation.

(a) $(f + g)(x)$

(b) $(f + g)(3)$

(c) $(fg)(x)$

(d) $(fg)(4)$

(e) $(f - g)(x)$

(f) $(f - g)(-2)$

(g) $\left(\frac{f}{g}\right)(x)$

Solution

(a)

$$(f+g)(x) = (x^2+1) + (3x-5) = x^2 + 3x - 4.$$

Both f and g are polynomials, so the domain is $(-\infty, \infty)$.

(b)

$$(f+g)(3) = 9 + 3(3) - 4 = 9 + 9 - 4 = 14.$$

(c)

$$(fg)(x) = (x^2+1)(3x-5) = 3x^3 - 5x^2 + 3x - 5.$$

Domain: $(-\infty, \infty)$.

(d)

$$(fg)(4) = (16+1)(12-5) = 17 \cdot 7 = 119.$$

(e)

$$(f-g)(x) = (x^2+1) - (3x-5) = x^2 - 3x + 6.$$

Domain: $(-\infty, \infty)$.

(f)

$$(f-g)(-2) = 4 - 3(-2) + 6 = 4 + 6 + 6 = 16.$$

(g)

$$\left(\frac{f}{g}\right)(x) = \frac{x^2+1}{3x-5}.$$

We need $3x - 5 \neq 0$, i.e., $x \neq \frac{5}{3}$. Domain: $\left(-\infty, \frac{5}{3}\right) \cup \left(\frac{5}{3}, \infty\right)$.

Example 2: If $f(x) = x^2 - 5$ and $g(x) = \sqrt{6-x}$, find $(fg)(x)$ and give the domain in interval notation.

Solution

$$(fg)(x) = (x^2 - 5)\sqrt{6 - x}.$$

f has domain $(-\infty, \infty)$. For g, we need $6 - x \geq 0$, i.e., $x \leq 6$. The combined domain is

$$(-\infty, 6].$$

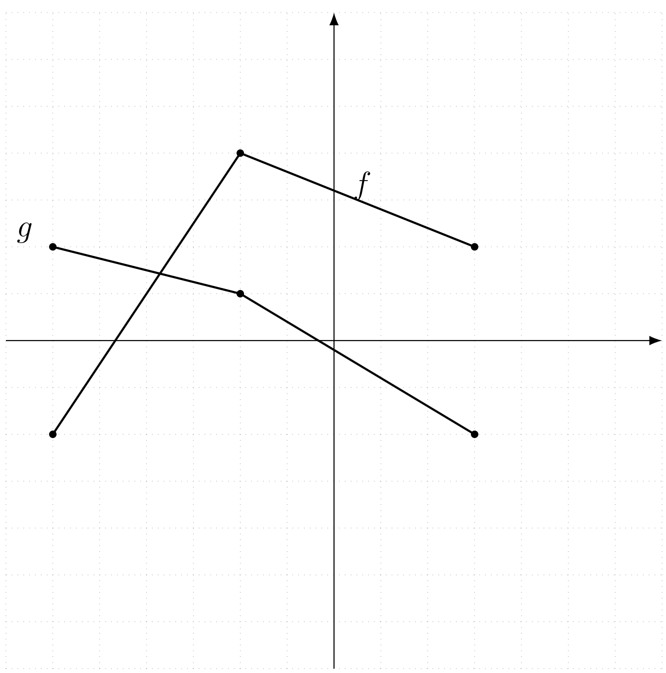
Example 3: Use the graph of f and g below to determine the following:

(a) $(f + g)(-6)$

(b) $(f - g)(-2)$

(c) $(f \cdot g)(3)$

(d) $\left(\frac{f}{g}\right)(-2)$



Solution

From the graph: $f(-6) = -2$, $f(-2) = 4$, $f(3) = 2$, and $g(-6) = 2$, $g(-2) = 1$, $g(3) = -2$.

(a)

$$(f+g)(-6) = f(-6) + g(-6) = -2 + 2 = 0.$$

(b)

$$(f-g)(-2) = f(-2) - g(-2) = 4 - 1 = 3.$$

(c)

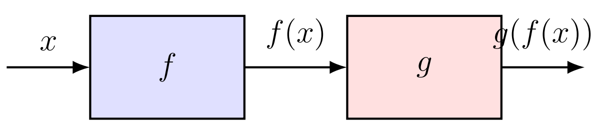
$$(f \cdot g)(3) = f(3) \cdot g(3) = 2 \cdot (-2) = -4.$$

(d)

$$\left(\frac{f}{g}\right)(-2) = \frac{f(-2)}{g(-2)} = \frac{4}{1} = 4.$$

2. Composition of Functions

Composing two functions is like connecting two machines that work one after the other: the output of the first machine becomes the input of the second.



$$(g \circ f)(x) = g(f(x))$$

Composition Notation

If we first apply f to an input x, and then apply g to the result, we write the composition as

$$(g \circ f)(x) = g(f(x)).$$

The order matters: $g(f(x))$ means f acts first and g acts second.

- $g(f(x))$: apply f's formula first, then apply g's formula to the result.
- $f(g(x))$: apply g's formula first, then apply f's formula to the result.

Important Distinction

$(f \circ g)(x)$ and $(fg)(x)$ are different functions: $(f \circ g)(x)$ is the composition, while $(fg)(x)$ is the product of $f(x)$ and $g(x)$. That is,

$$f \circ g \neq fg.$$

Example 4: Given $f(x) = 3x + 4$ and $g(x) = \dfrac{1}{x-1}$, find the following and write the domain in interval notation.

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

Solution

(a)

$$(f \circ g)(x) = f(g(x)) = 3\left(\frac{1}{x-1}\right) + 4 = \frac{3}{x-1} + 4 = \frac{3+4(x-1)}{x-1} = \frac{4x-1}{x-1}.$$

We need $x \neq 1$. Domain: $(-\infty, 1) \cup (1, \infty)$.
(b)

$$(g \circ f)(x) = g(f(x)) = \frac{1}{(3x+4)-1} = \frac{1}{3x+3} = \frac{1}{3(x+1)}.$$

We need $x \neq -1$. Domain: $(-\infty, -1) \cup (-1, \infty)$.

Example 5: $f(x) = 4 - x$ and $g(x) = 2x + 1$, find:

(a) $g(f(1))$

(b) $f(g(3))$

(c) Compute $f(g(2))$ and $g(f(2))$. Are the two values equal?

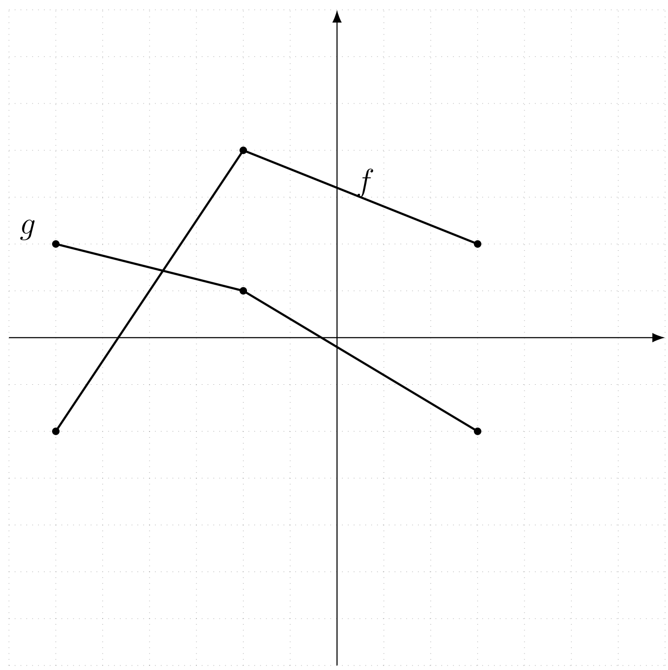
Solution

(a) $f(1) = 4 - 1 = 3$, so $g(f(1)) = g(3) = 2(3) + 1 = 7$.
(b) $g(3) = 2(3) + 1 = 7$, so $f(g(3)) = f(7) = 4 - 7 = -3$.
(c) $g(2) = 2(2) + 1 = 5$, so $f(g(2)) = f(5) = 4 - 5 = -1$.
$f(2) = 4 - 2 = 2$, so $g(f(2)) = g(2) = 2(2) + 1 = 5$.
Since $-1 \neq 5$, the two values are not equal --- in general, $f \circ g \neq g \circ f$.

Example 6: Use the graph of f and g below to compute the following.

(a) $(g \circ f)(-6)$

(b) $(f \circ g)(3)$



Solution

As before: $f(-6) = -2$, $f(-2) = 4$, $f(3) = 2$, and $g(-6) = 2$, $g(-2) = 1$, $g(3) = -2$.

(a)

$$(g \circ f)(-6) = g(f(-6)) = g(-2) = 1.$$

(b)

$$(f \circ g)(3) = f(g(3)) = f(-2) = 4.$$

Example 7: Using the table below, find the following.

x	0	2	3	4	5
$f(x)$	3	1	2	5	0
$g(x)$	5	7	4	3	1

(a) $(g-f)(2)$

(b) $(f+g)(0)$

(c) $(fg)(3)$

(d) $\left(\frac{g}{f}\right)(5)$

(e) $(g \circ f)(4)$

(f) $(f \circ g)(4)$

(g) $(f \circ g)(2)$

Solution

(a) $(g-f)(2) = g(2) - f(2) = 7 - 1 = 6$.
(b) $(f+g)(0) = f(0) + g(0) = 3 + 5 = 8$.
(c) $(fg)(3) = f(3) \cdot g(3) = 2 \cdot 4 = 8$.
(d) $\left(\frac{g}{f}\right)(5) = \frac{g(5)}{f(5)} = \frac{1}{0}$, which is undefined --- $f(5) = 0$, so we cannot divide by it.
(e) $(g \circ f)(4) = g(f(4)) = g(5) = 1$.
(f) $(f \circ g)(4) = f(g(4)) = f(3) = 2$.
(g) $(f \circ g)(2) = f(g(2)) = f(7)$. But $x = 7$ does not appear in the table, so $f(7)$ is not determined by the given information --- this composition cannot be computed from the table.

Example 8: Suppose $f(x)$ gives the miles that can be driven in x hours, and $g(y)$ gives the gallons of gas used driving y miles. What does the composite function $(g \circ f)(x)$ give? Is $f \circ g$ a meaningful composition? Explain why.

Solution

$(g \circ f)(x) = g(f(x))$: first find $f(x)$, the miles driven in x hours; then apply g to that many miles, giving the gallons of gas used. So $(g \circ f)(x)$ gives the gallons of gas used after driving for x hours. This is meaningful, since the output of f (miles) matches exactly what g expects as an input (miles).

$f \circ g$ is not meaningful: $g(y)$ outputs gallons of gas, but f expects an input in hours. Feeding a number of gallons into f does not correspond to any real quantity --- the output type of g does not match the input type of f.

Summary

Key Ideas

- $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$ combine f and g pointwise; their domain is the overlap of the domains of f and g.
- $\left(\frac{f}{g}\right)(x)$ also requires $g(x) \neq 0$.
- $(g \circ f)(x) = g(f(x))$ means f acts first, then g acts on the result. Composition order matters: usually $f \circ g \neq g \circ f$.
- $(f \circ g)(x)$ and $(fg)(x)$ are different operations --- composition versus product.
- A composition is only meaningful when the output type of the inner function matches the expected input of the outer function.