Module 2 Lesson 5
Transformations of Graphs
(Building new graphs from old)

Objectives

In this lesson, we will learn how to

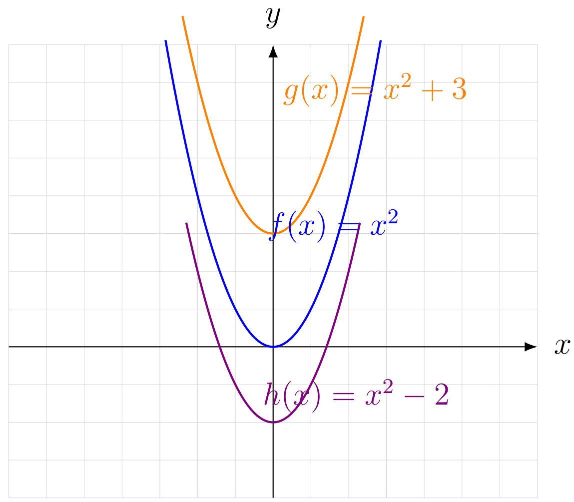
- shift a graph vertically and horizontally;
- reflect a graph about the x-axis and the y-axis;
- stretch or shrink a graph vertically and horizontally;
- describe, and build a formula for, a combination of transformations.

1. Vertical Shifts

Vertical Shifts

Adding or subtracting a constant from a function shifts its graph up or down. If k is a positive constant, then

- $f(x) + k$ shifts the graph of the function up by k units;
- $f(x) - k$ shifts the graph of the function down by k units.

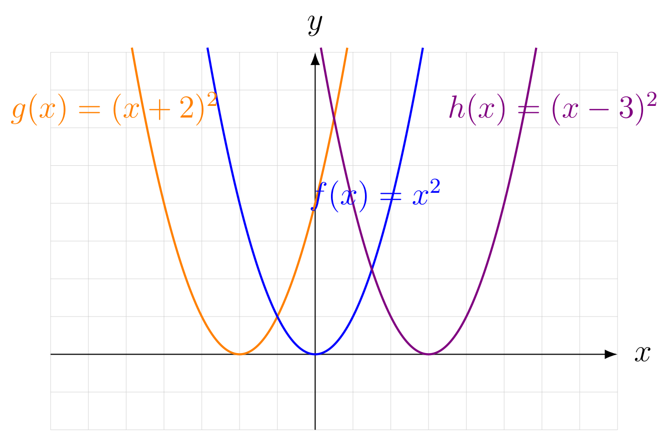


2. Horizontal Shifts

Horizontal Shifts

Replacing x by $x + h$ or $x - h$ shifts the graph left or right. If h is a positive constant, then

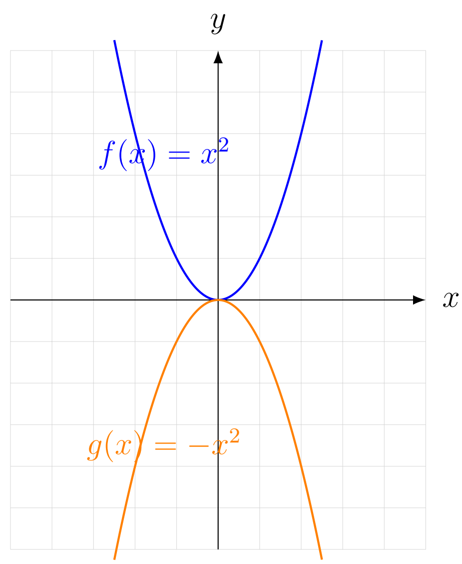
- $f(x - h)$ shifts the graph of the function right by h units;
- $f(x + h)$ shifts the graph of the function left by h units.



3. Reflecting About the x-axis

Reflecting About the x-axis

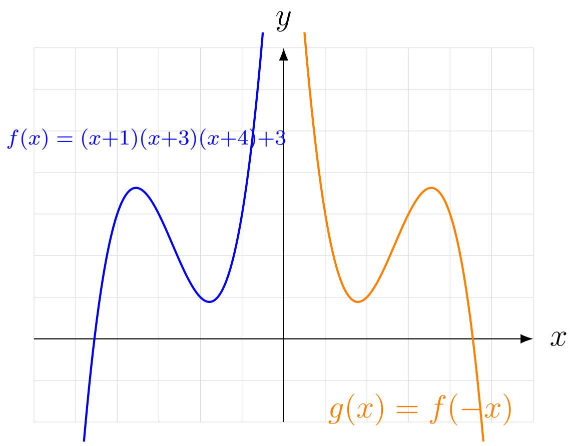
Multiplying a function by -1 reflects the graph about the x-axis.



4. Reflecting About the y-axis

Reflecting About the y-axis

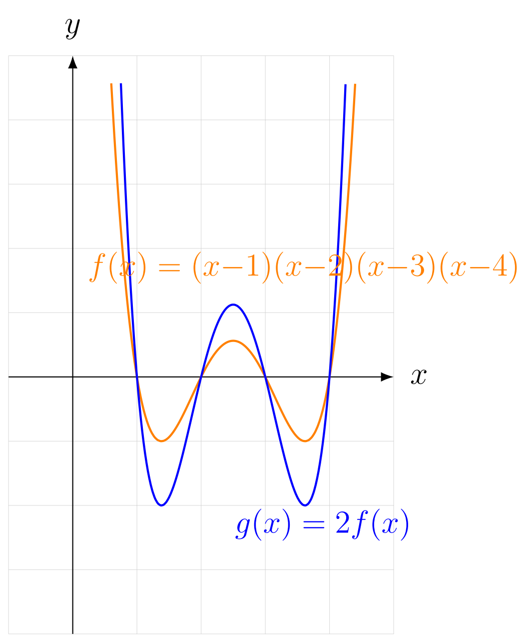
Replacing x by $-x$ in the formula for a function reflects the graph about the y-axis.



5. Vertical Stretching/Shrinking

Vertical Stretching/Shrinking

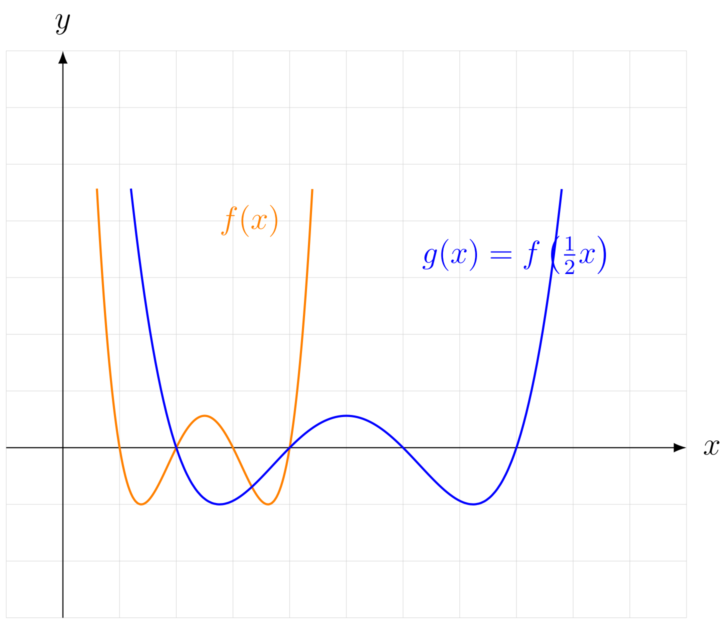
Multiplying a function by a factor of a stretches (if $a > 1$) or shrinks (if $0 < a < 1$) the graph of the function vertically by a factor of a.



6. Horizontal Stretching/Shrinking

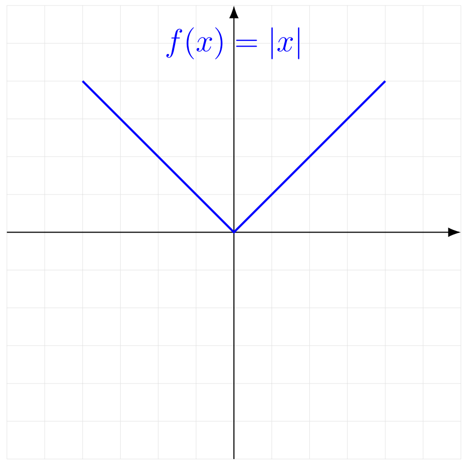
Horizontal Stretching/Shrinking

Multiplying x by a factor of a shrinks (if $0 < a < 1$) or stretches (if $a > 1$) the graph of the function horizontally by a factor of a.



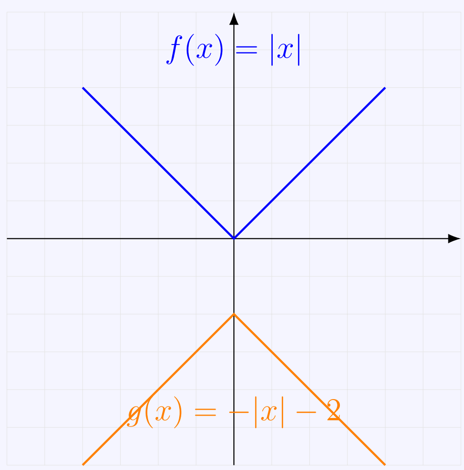
7. Practice: Sketching Transformations

Example 1: Given the graph of $f(x) = |x|$, sketch the graph of $g(x) = -f(x) - 2 = -|x| - 2$.

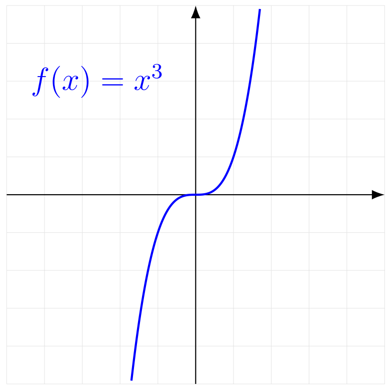


Solution

$g(x) = -|x| - 2$ reflects f about the x-axis (the sign flips) and then shifts it down 2 units. The vertex moves from $(0,0)$ to $(0,-2)$, and the ``V'' now opens downward.

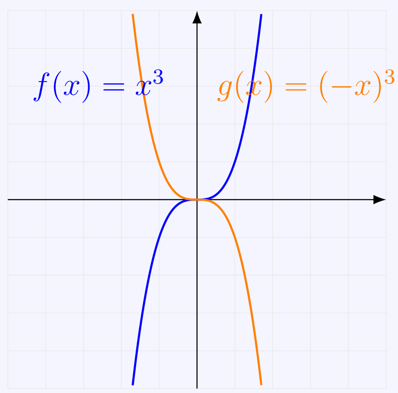


Example 2: Given the graph of $f(x) = x^3$, sketch the graph of $g(x) = f(-x) = (-x)^3$.

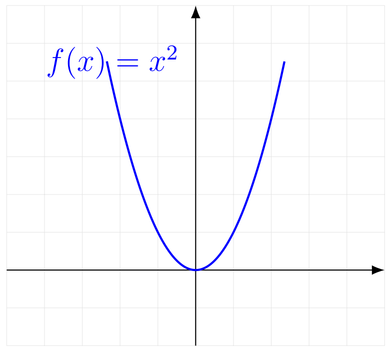


Solution

Since $(-x)^3 = -x^3$, we have $g(x) = -x^3$. Reflecting x^3 about the y-axis gives the same picture as reflecting it about the x-axis --- this is a special feature of odd functions like x^3.

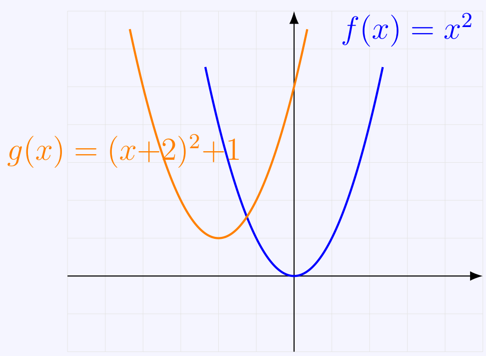


Example 3: Given the graph of $f(x) = x^2$, sketch the graph of $g(x) = (x+2)^2 + 1$.

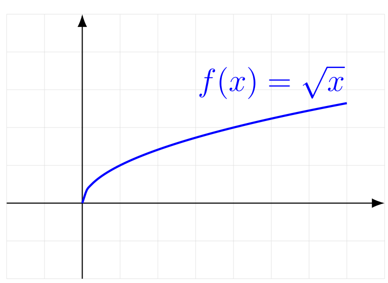


Solution

$g(x) = (x+2)^2+1$ shifts f left 2 units and up 1 unit. The vertex moves from $(0,0)$ to $(-2,1)$.

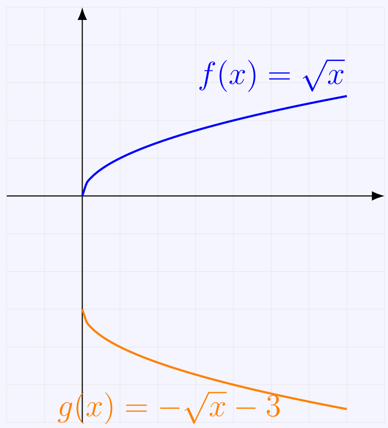


Example 4: Starting with the given graph of $f(x) = \sqrt{x}$, sketch the graph of $g(x) = -\sqrt{x} - 3$.



Solution

$g(x) = -\sqrt{x} - 3$ reflects f about the x-axis and then shifts it down 3 units.



8. Describing Combinations of Transformations

Example 5: List all the transformations that have been applied to $y = f(x)$ to obtain $y = \dfrac{1}{4}f(x+2) - 5$.

Solution

Reading from the inside out:

- $f(x+2)$: shift left 2 units;
- $\dfrac{1}{4}f(x+2)$: vertically shrink by a factor of $\dfrac{1}{4}$;
- $\dfrac{1}{4}f(x+2) - 5$: shift down 5 units.

Example 6: List all the transformations that have been applied to $y = f(x)$ to obtain $g(x) = 3f(-x + 1) + 2$.

Solution

Rewrite the argument: $-x+1 = -(x-1)$, so $g(x) = 3f(-(x-1))+2$.

- $f(-x)$: reflect about the y-axis;
- $f(-(x-1))$: shift right 1 unit;
- $3f(-(x-1))$: vertically stretch by a factor of 3;
- $3f(-(x-1)) + 2$: shift up 2 units.

Example 7: Describe all the transformations that have been applied to $y = f(x)$ to obtain $y = -2f(-(x + 4)) - 7$.

Solution

- $f(-x)$: reflect about the y-axis;
- $f(-(x+4))$: shift left 4 units;
- $-2f(-(x+4))$: vertically stretch by a factor of 2 and reflect about the x-axis;
- $-2f(-(x+4)) - 7$: shift down 7 units.

Example 8: Write a formula for the function obtained when the graph of $f(x) = \sqrt{x}$ is shifted up 2 units, shifted left 3 units, and reflected across the x-axis.

Solution

Apply the transformations in the order listed:

$$\sqrt{x} \xrightarrow{\text{up } 2} \sqrt{x}+2 \xrightarrow{\text{left } 3} \sqrt{x+3}+2 \xrightarrow{\text{reflect across } x\text{-axis}} -\left(\sqrt{x+3}+2\right).$$

$$g(x) = -\sqrt{x+3} - 2.$$

Example 9: Write a formula for the function obtained when the graph of $f(x) = \dfrac{1}{\sin(x)}$ is stretched horizontally by 3, reflected across the y-axis, and moved down 2 units.

Solution

Apply the transformations in the order listed:

$$\frac{1}{\sin x} \xrightarrow{\text{stretch horiz. by } 3} \frac{1}{\sin(x/3)} \xrightarrow{\text{reflect across } y\text{-axis}} \frac{1}{\sin(-x/3)},$$

$$\frac{1}{\sin(-x/3)} \xrightarrow{\text{down } 2} \frac{1}{\sin(-x/3)} - 2.$$

Since $\sin(-x/3) = -\sin(x/3)$, this simplifies to

$$g(x) = -\frac{1}{\sin(x/3)} - 2.$$

Summary

Key Ideas

- $f(x) + k$ / $f(x) - k$: shift up/down by k.
- $f(x - h)$ / $f(x + h)$: shift right/left by h.
- $-f(x)$: reflect about the x-axis. $f(-x)$: reflect about the y-axis.
- $a \cdot f(x)$: vertical stretch ($a > 1$) or shrink ($0 < a < 1$).
- $f(ax)$: horizontal shrink ($a > 1$) or stretch ($0 < a < 1$).
- When combining transformations, work from the inside of the formula outward, and build new formulas by applying each transformation to the previous result, in the order given.