Module 3 Lesson 1
Introduction to Polynomial Functions

Objectives

In this lesson, we will learn how to

- identify polynomial functions and their degree;
- determine end behavior using the leading-term test;
- find zeros of a polynomial and their multiplicity;
- classify zeros as cross or bounce points;
- find x- and y-intercepts of a polynomial function.

1. Polynomial Functions

Polynomial Function

A function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0, \qquad a_n \neq 0,$$

is a polynomial function.

Examples:

$$f(x) = 2x^3 - 4x^2 + 5x - 6, \qquad f(x) = -3x^5 + 7x - 9,$$

$$y = 4x^6 - 7x^2 - 6x^3 - 4x^4 + 5x - 6, \qquad V = \frac{4}{3}\pi r^3.$$

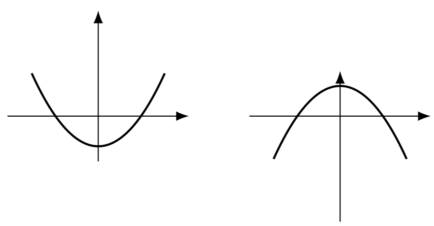
- The coefficients $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are real numbers. The powers $n, n-1, n-2, \ldots$ are non-negative integers (whole numbers).
- The degree of a polynomial function is the highest power of x in the formula.
- The graph of a polynomial function is both continuous and smooth. Informally, a continuous function can be drawn without lifting the pencil from the paper, and a smooth function has no sharp corners or points.

2. Turning Points

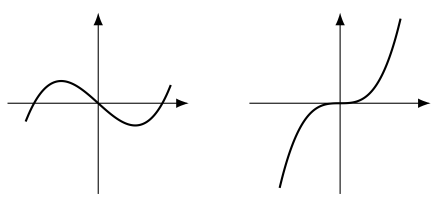
Turning Points

A polynomial function of degree n has at most $n-1$ hills or valleys (turning points).

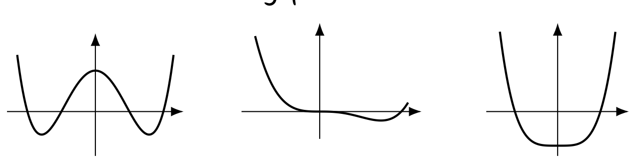
Degree 2 (quadratic) --- 1 turning point



Degree 3 (cubic) --- at most 2 turning points



Degree 4 --- at most 3 turning points



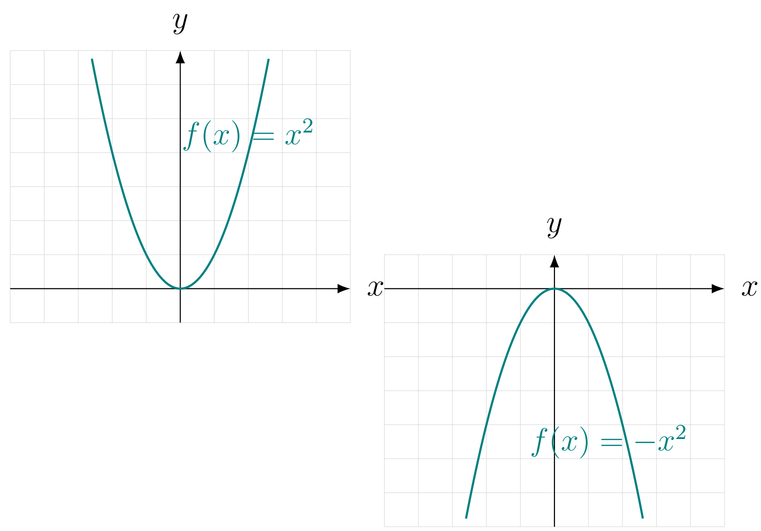
3 turning points 2 turning points 1 turning point

3. End Behavior (Leading-Term Test)

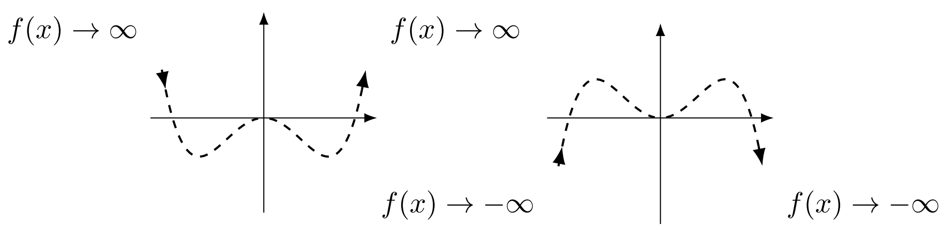
The left-hand/right-hand end behavior is determined by (1) whether the degree is even or odd, and (2) the sign of the leading coefficient.

Even Degree. The ends shoot off in the same direction.

- If the leading coefficient is positive, both ends go up.
- If the leading coefficient is negative, both ends go down.

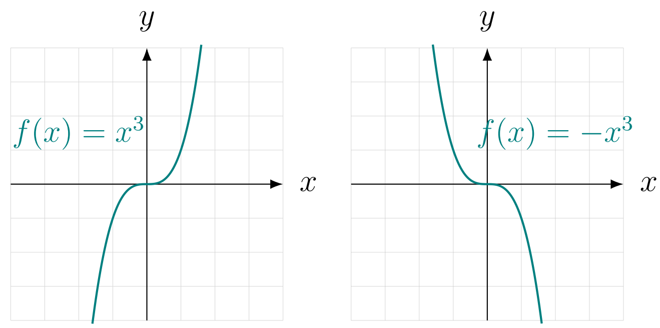


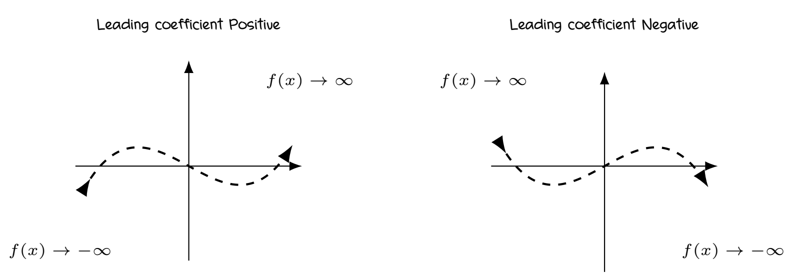
Leading coefficient Positive   Leading coefficient Negative



Odd Degree. The ends shoot off in opposite directions.

- If the leading coefficient is positive, left down and right up.
- If the leading coefficient is negative, left up and right down.





Example 1: Determine the end behavior of each polynomial.

(a) $f(x) = 3x^4 - 5x^2 + 1$

(b) $f(x) = -2x^5 + 3x^3 - x$

(c) $f(x) = -4x^2 + 7x - 1$

(d) $f(x) = 6x^3 - x^2 + 2$

(e) $f(x) = 5x^6 - 2x + 9$

Solution

(a) Degree 4 (even), leading coefficient $3 > 0$: both ends go up ($f(x) \to \infty$ as $x \to \pm\infty$).

(b) Degree 5 (odd), leading coefficient $-2 < 0$: left up, right down ($f(x) \to \infty$ as $x \to -\infty$, $f(x) \to -\infty$ as $x \to \infty$).

(c) Degree 2 (even), leading coefficient $-4 < 0$: both ends go down ($f(x) \to -\infty$ as $x \to \pm\infty$).

(d) Degree 3 (odd), leading coefficient $6 > 0$: left down, right up.

(e) Degree 6 (even), leading coefficient $5 > 0$: both ends go up.

4. Zeros and Multiplicity

Zeros of a Polynomial

A zero of a polynomial function f is a value c such that $f(c) = 0$. If $(x - c)^m$ is a factor of $f(x)$ and $(x - c)^{m+1}$ is not, then c is a zero of multiplicity m.

Example 2: List the zeros of each polynomial and state their multiplicity.

(a) $f(x) = (x - 2)^3(x + 1)^2$

(b) $f(x) = x(x - 4)(x + 5)^4$

(c) $f(x) = -2(x - 1)(x - 3)^2(x + 2)^5$

Solution

(a) $x = 2$, multiplicity 3 (odd); $x = -1$, multiplicity 2 (even).

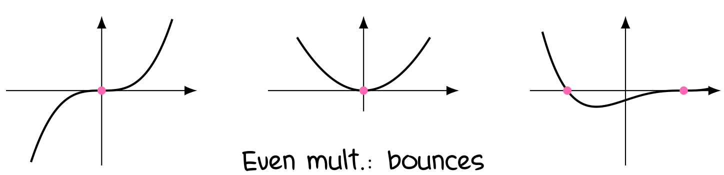
(b) $x = 0$, multiplicity 1 (odd);
$x = 4$, multiplicity 1 (odd);
$x = -5$, multiplicity 4 (even).

(c) $x = 1$, multiplicity 1 (odd);
$x = 3$, multiplicity 2 (even);
$x = -2$, multiplicity 5 (odd).

5. Cross vs. Bounce Points

Behavior at a Zero

- If the multiplicity m is odd, the graph crosses the x-axis at c.
- If the multiplicity m is even, the graph touches the x-axis and bounces back (turns around) at c.



Even mult.: bounces

Odd mult.: crosses

Mult. 3 and mult. 1: both cross

Example 3: Find the zeros of each polynomial and classify each as a cross point or bounce point.

(a) $f(x) = x^4 + 4x^3 + 3x^2$

(b) $f(x) = x^3 - 5x^2 - x + 5$

Solution

(a) Factor out the GCF, then factor the remaining trinomial:

$$f(x) = x^2(x^2 + 4x + 3) = x^2(x + 1)(x + 3).$$

Zeros: $x = 0$ (mult. 2, bounce), $x = -1$ (mult. 1, cross), $x = -3$ (mult. 1, cross).
(b) Factor by grouping:

$$f(x) = x^2(x - 5) - 1(x - 5) = (x - 5)(x^2 - 1) = (x - 5)(x - 1)(x + 1).$$

Zeros: $x = 5,\ x = 1,\ x = -1$, each multiplicity 1 (all cross points).

6. Intercepts

Finding Intercepts

- y-intercept: evaluate $f(0)$.
- x-intercepts: solve $f(x) = 0$ (these are the zeros of f).

Example 4: Find the x- and y-intercepts of $f(x) = (x - 2)(x + 1)(x + 4)$.

Solution

x-intercepts (zeros): $x = 2,\ x = -1,\ x = -4$, i.e. $(2, 0),\ (-1, 0),\ (-4, 0)$.
y-intercept: $f(0) = (0-2)(0+1)(0+4) = (-2)(1)(4) = -8$, i.e. $(0, -8)$.

7. Reasoning About Degree

Example 5: A polynomial's graph has exactly 2 turning points and both ends rise. What is the smallest possible degree, and is the leading coefficient positive or negative?

Solution

A degree-n polynomial has at most $n-1$ turning points, so 2 turning points requires degree at least 3. Since both ends rise, the degree must be even, so the smallest possible degree is $\boxed{4}$. Both ends rising means the leading coefficient is positive.

Example 6: A polynomial's graph crosses the x-axis at $x=-3,0,2$ and has y-intercept 0. Both ends of the graph fall. What is the smallest possible degree?

Solution

Three distinct crossing zeros require at least degree 3 (odd). But both ends falling means the same direction on both sides, which requires an even degree. The smallest even degree that can still produce 3 sign changes (crossings) is $\boxed{4}$ --- one of the zeros would need even multiplicity, e.g. $f(x)=-(x+3)(x)(x-2)^2$, or another zero is repeated so total multiplicity is 4.

Example 7: The graph of a polynomial touches the x-axis at $x=1$ without crossing, and crosses at $x=-2$. What is the smallest possible degree?

Solution

Touching (bounce) at $x=1$ requires even multiplicity, smallest is 2. Crossing at $x=-2$ requires odd multiplicity, smallest is 1. Total smallest degree: $2+1=\boxed{3}$.

Summary

Key Ideas

- A polynomial function has the form $a_n x^n + \cdots + a_1 x + a_0$; its degree is the highest power of x, and its graph is always continuous and smooth.
- A degree-n polynomial has at most $n-1$ turning points.
- End behavior is determined by the degree's parity and the leading coefficient's sign: even degree $\to$ both ends the same direction; odd degree $\to$ opposite directions; positive leading coefficient $\to$ right end up; negative $\to$ right end down.
- If $(x-c)^m$ is a factor of $f(x)$, then c is a zero of multiplicity m: odd multiplicity $\to$ the graph crosses the x-axis; even multiplicity $\to$ the graph bounces.
- The y-intercept is $f(0)$; the x-intercepts are the zeros of f.