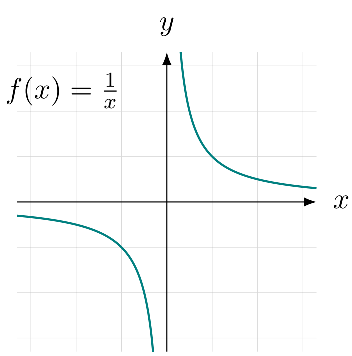
Module 3 Lesson 3A
Graphing Rational Functions

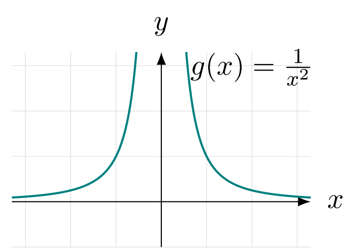
Objectives

In this lesson, we will learn how to

- graph the toolkit functions $\dfrac{1}{x}$ and $\dfrac{1}{x^2}$;
- graph transformations of these toolkit functions;
- fully graph a rational function using domain, intercepts, asymptotes, holes, and a table of points.

1. Toolkit Functions





Key Features

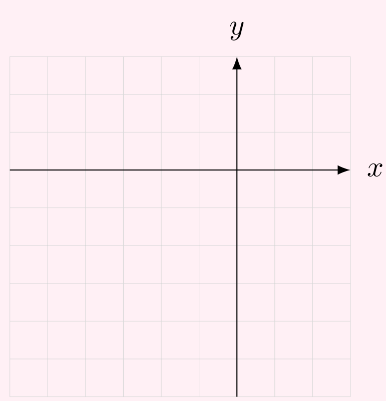
- Both toolkit functions have a vertical asymptote at $x = 0$ and horizontal asymptote $y = 0$.
- $f(x) = \frac{1}{x}$ is an odd function (symmetric about the origin); its two branches lie in Quadrants I and III.
- $g(x) = \frac{1}{x^2}$ is an even function (symmetric about the y-axis); both branches lie above the x-axis, in Quadrants I and II.

2. Graphing Transformations

Transforming $\frac{1}{x}$ or $\frac{1}{x^2}$

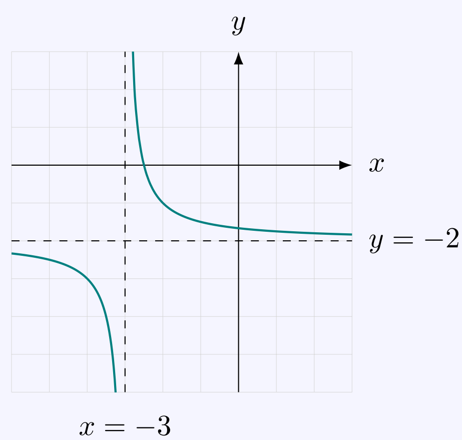
Writing a rational function as $\dfrac{1}{x-h} + k$ (or $\dfrac{1}{(x-h)^2} + k$) shifts the toolkit graph so that the vertical asymptote is $x = h$ and the horizontal asymptote is $y = k$.

Example 1: Sketch $h(x) = \dfrac{1}{x+3} - 2$.

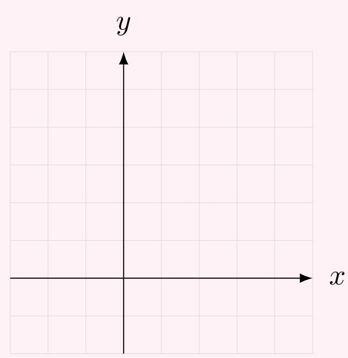


Solution

This is $\frac{1}{x}$ shifted left 3 and down 2. Vertical asymptote: $x = -3$. Horizontal asymptote: $y = -2$.

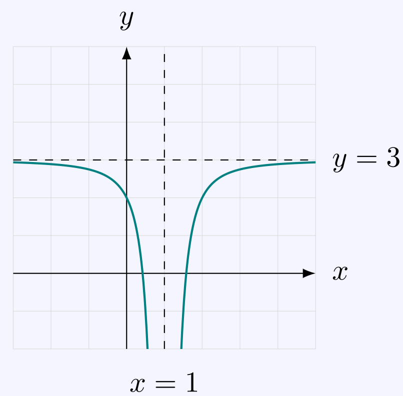


Example 2: Sketch $k(x) = -\dfrac{1}{(x-1)^2} + 3$.



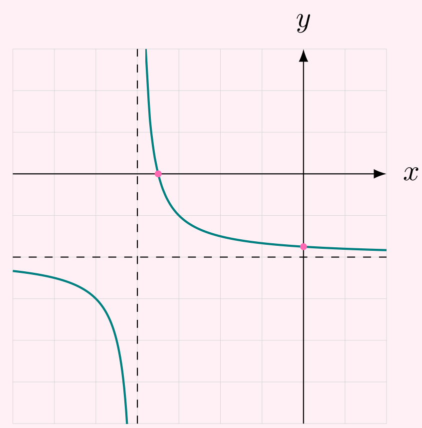
Solution

This is $\dfrac{1}{x^2}$ shifted right 1, reflected over the x-axis, and shifted up 3. Vertical asymptote: $x = 1$. Horizontal asymptote: $y = 3$.



3. Reading Features from a Graph

Example 3: Use the graph below to identify the vertical asymptote, horizontal asymptote, x-intercept, and y-intercept.



Solution

Vertical asymptote: $x = -4$. Horizontal asymptote: $y = -2$.
x-intercept: $(-3.5, 0)$. y-intercept: $(0, -1.75)$.
(This graph matches $f(x) = \dfrac{1}{x+4} - 2$: setting $f(x) = 0$ gives $x = -3.5$, and $f(0) = \frac{1}{4} - 2 = -1.75$.)

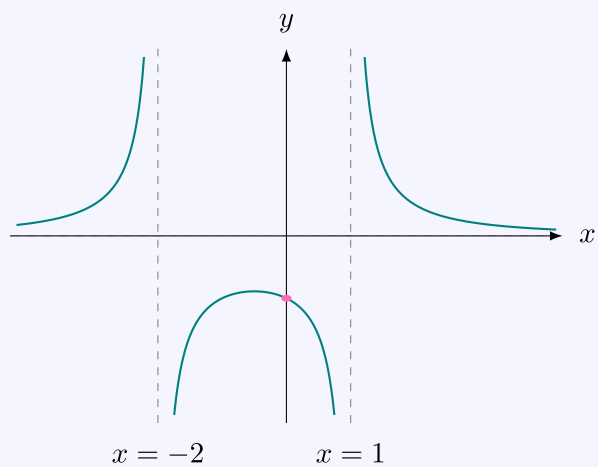
4. Full Worked Examples

Example 4: Graph $f(x) = \dfrac{3}{(x+2)(x-1)}$.

Solution

Domain: $x \neq -2, 1$.
Vertical asymptotes: $x = -2,\ x = 1$ (no common factors, so no holes).
Horizontal asymptote: numerator degree $0 <$ denominator degree 2, so $y = 0$.
y-intercept: $f(0) = \dfrac{3}{(2)(-1)} = -1.5$. x-intercepts: none (numerator is never 0).

x	-4	-3	0.5	2	3
$f(x)$	0.3	0.75	-2.4	0.75	0.3



Example 5: Graph $f(x) = \dfrac{x+1}{x^2+2x-3}$.

Solution

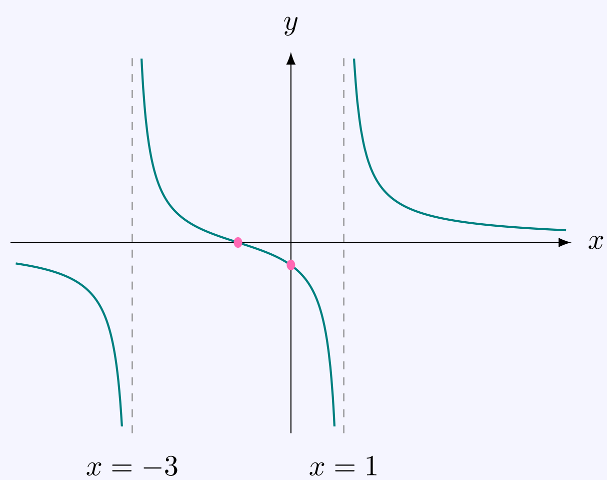
Factor: $f(x) = \dfrac{x+1}{(x+3)(x-1)}$. The numerator shares no common factor with the denominator, so there are no holes.

Domain: $x \neq -3, 1$. Vertical asymptotes: $x = -3,\ x = 1$.

Horizontal asymptote: numerator degree $1 <$ denominator degree 2, so $y = 0$ (and there is no slant asymptote, since the numerator's degree is not exactly one more than the denominator's).

y-intercept: $f(0) = \dfrac{1}{-3} = -0.33$. x-intercept: $x + 1 = 0 \implies x = -1$, i.e. $(-1, 0)$.

x	-4	-2	0.5	2	3
$f(x)$	0.6	0.33	-0.86	0.6	0.33



Example 6: Graph $f(x) = \dfrac{x^2 - x - 2}{x^2 - 4}$.

Solution

Factor: $f(x) = \dfrac{(x-2)(x+1)}{(x-2)(x+2)} = \dfrac{x+1}{x+2}$ for $x \neq 2$.

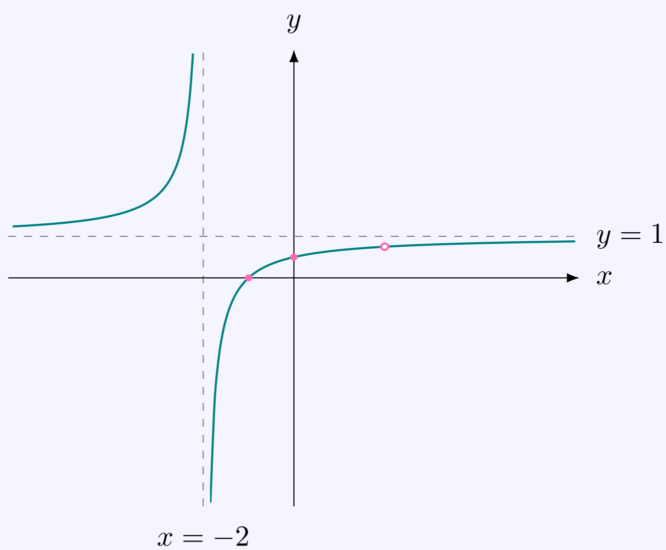
Hole: at $x = 2$. Hole's y-value: $\dfrac{2+1}{2+2} = \dfrac{3}{4} = 0.75$, so the hole is at $(2, 0.75)$.

Domain: $x \neq -2, 2$. Vertical asymptote: $x = -2$.

Horizontal asymptote: numerator and denominator both degree 1 (after simplifying), ratio $\dfrac{1}{1} = 1$, so $y = 1$.

y-intercept: $f(0) = \dfrac{1}{2} = 0.5$. x-intercept: $x + 1 = 0 \implies x = -1$, i.e. $(-1, 0)$.

x	-4	-3	-1.5	1	3
$f(x)$	1.5	2	-1	0.67	0.8



Summary

Key Ideas

- The toolkit functions $\frac{1}{x}$ and $\frac{1}{x^2}$ both have vertical asymptote $x = 0$ and horizontal asymptote $y = 0$; $\frac{1}{x}$ is odd (Quadrants I, III), while $\frac{1}{x^2}$ is even (Quadrants I, II).
- Writing a rational function as $\frac{1}{x-h} + k$ (or $\frac{1}{(x-h)^2} + k$) shifts the toolkit graph so the vertical asymptote is $x = h$ and the horizontal asymptote is $y = k$.
- A full graph combines: domain, holes, vertical/horizontal asymptotes, x- and y-intercepts, and a table of additional points to confirm the shape of each branch.