Module 3 Lesson 3
Rational Functions

Objectives

In this lesson, we will learn how to

- find the domain of a rational function;
- distinguish a vertical asymptote from a removable hole;
- find horizontal asymptotes using the degree of the numerator and denominator;
- find where a graph crosses its horizontal asymptote.

1. Domain of a Rational Function

Rational Function

A rational function has the form $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. The domain excludes every x for which $q(x) = 0$.

Example 1: Find the domain of each rational function.

(a) $f(x) = \dfrac{x+3}{x^2-9}$

(b) $g(x) = \dfrac{2x}{x^2+x-6}$

Solution

(a) $x^2 - 9 = (x-3)(x+3) = 0 \implies x = 3, -3$. Domain: all reals except $x = 3, -3$, i.e. $\{x \mid x \neq \pm 3\}$.
(b) $x^2 + x - 6 = (x+3)(x-2) = 0 \implies x = -3, 2$. Domain: all reals except $x = -3, 2$.

2. Vertical Asymptotes vs. Holes

Finding Vertical Asymptotes and Holes

Factor the numerator and denominator completely.

- If a factor $(x-c)$ cancels from both numerator and denominator, the graph has a hole at $x = c$ (a single missing point).
- If a factor $(x-c)$ remains in the denominator after cancelling, the graph has a vertical asymptote at $x = c$.

Example 2: For each function, identify any holes and vertical asymptotes.

(a) $f(x) = \dfrac{x-2}{x^2-4}$

(b) $g(x) = \dfrac{x+1}{x^2-1}$

(c) $h(x) = \dfrac{x^2-4}{x^2+5x+6}$

Solution

(a) $f(x) = \dfrac{x-2}{(x-2)(x+2)} = \dfrac{1}{x+2}$ (for $x \neq 2$).
Hole at $x = 2$; vertical asymptote at $x = -2$.
(b) $g(x) = \dfrac{x+1}{(x+1)(x-1)} = \dfrac{1}{x-1}$ (for $x \neq -1$).
Hole at $x = -1$; vertical asymptote at $x = 1$.
(c) $h(x) = \dfrac{(x-2)(x+2)}{(x+2)(x+3)} = \dfrac{x-2}{x+3}$ (for $x \neq -2$).
Hole at $x = -2$; vertical asymptote at $x = -3$.

3. Horizontal Asymptotes

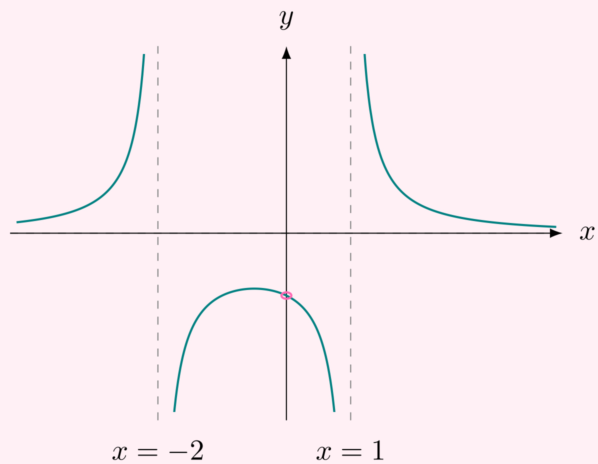
Horizontal Asymptote Rules

Compare the degree of the numerator n to the degree of the denominator m.

- If $n < m$: horizontal asymptote is $y = 0$.
- If $n = m$: horizontal asymptote is $y = \dfrac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}}$.
- If $n > m$: there is no horizontal asymptote (the function may have a slant asymptote instead).

4. Reading Graphs of Rational Functions

Example 3: The graph of $f(x) = \dfrac{3x}{x(x+2)(x-1)}$ is shown below. Identify the vertical asymptotes, the horizontal asymptote, and any holes.



Solution

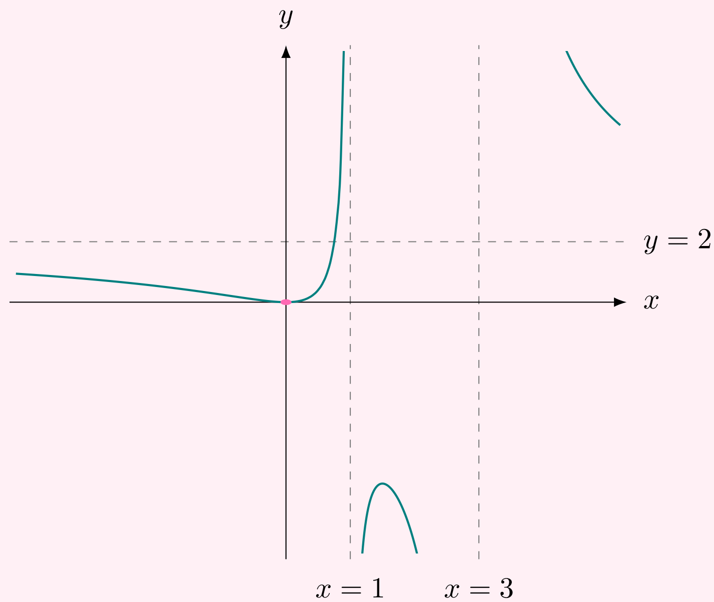
Simplify first: $f(x) = \dfrac{3x}{x(x+2)(x-1)} = \dfrac{3}{(x+2)(x-1)}$ for $x \neq 0$.

Vertical asymptotes: $x = -2$ and $x = 1$ (from the remaining denominator factors).

Hole: at $x = 0$ (the factor x cancels). The hole's y-value: $\dfrac{3}{(0+2)(0-1)} = \dfrac{3}{-2} = -1.5$, so the hole is at $(0, -1.5)$ --- notice this means the graph has no y-intercept, even though $x = 0$ looks like it should give one!

Horizontal asymptote: numerator degree $0 <$ denominator degree 2 (after simplifying), so $y = 0$.

Example 4: The graph of $f(x) = \dfrac{2x^2}{x^2 - 4x + 3}$ is shown below. Identify the vertical asymptotes, the horizontal asymptote, and the intercepts.



Solution

No common factors between $2x^2$ and $x^2 - 4x + 3 = (x-1)(x-3)$, so there are no holes.
Vertical asymptotes: $x = 1$ and $x = 3$.
Horizontal asymptote: numerator and denominator both degree 2, ratio of leading coefficients $\dfrac{2}{1} = 2$, so $y = 2$.
Intercepts: $f(0) = \dfrac{0}{3} = 0$, so $(0, 0)$ is both the x- and y-intercept. Since $2x^2$ has a double zero at $x = 0$, the graph touches the x-axis at the origin rather than crossing it.

5. Crossing the Horizontal Asymptote

A Common Misconception

A graph can never cross a vertical asymptote, but it can cross a horizontal asymptote. To find where, set $f(x)$ equal to the horizontal asymptote's y-value and solve.

Example 5: Find the horizontal asymptote of $f(x) = \dfrac{x+3}{x^2+1}$, and determine where the graph crosses it.

Solution

Numerator degree $1 <$ denominator degree 2, so the horizontal asymptote is $y = 0$.
Set $f(x) = 0$:

$$\frac{x+3}{x^2+1} = 0 \implies x+3=0 \implies x=-3.$$

The graph crosses its horizontal asymptote at $\boxed{(-3,0)}$.

Example 6: Find the horizontal asymptote of $g(x) = \dfrac{-2x^2+x-4}{x^2+1}$, and determine where the graph crosses it.

Solution

Numerator and denominator both degree 2, ratio of leading coefficients $\dfrac{-2}{1} = -2$, so the horizontal asymptote is $y = -2$.
Set $g(x) = -2$:

$$\frac{-2x^2 + x - 4}{x^2 + 1} = -2 \implies -2x^2 + x - 4 = -2(x^2 + 1) = -2x^2 - 2.$$

The $-2x^2$ terms cancel:

$$x - 4 = -2 \implies x = 2.$$

The graph crosses its horizontal asymptote at $\boxed{(2, -2)}$.

Summary

Key Ideas

- The domain of a rational function excludes every x that makes the denominator 0.
- Factor the numerator and denominator completely: a factor that cancels produces a hole; a factor that remains in the denominator produces a vertical asymptote.
- Horizontal asymptotes depend on the degrees of the numerator (n) and denominator (m): $n < m \Rightarrow y = 0$; $n = m \Rightarrow y =$ ratio of leading coefficients; $n > m \Rightarrow$ no horizontal asymptote.
- A graph can never cross a vertical asymptote, but it can cross a horizontal asymptote --- set $f(x)$ equal to the asymptote's y-value and solve.