Module 3 Lesson 5
Inverse Functions

Objectives

In this lesson, we will learn how to

- find the inverse of a relation;
- determine whether a function is one-to-one and has an inverse;
- verify that two functions are inverses of each other;
- find the inverse of a function algebraically;
- evaluate and sketch inverse functions from a graph or table.

1. Inverse Relations

Inverse Relation

Interchanging the first and second coordinate of each ordered pair in a relation produces the inverse relation. If a relation is defined by an equation, interchanging the variables produces an equation of the inverse relation.

Example 1: Find the inverse relation.

(a) Relation: $\{(3,2),(4,1),(5,2)\}$

(b) Relation: $\{(1,2),(3,4),(5,6)\}$

(c) Relation: $y = 3x - 5$

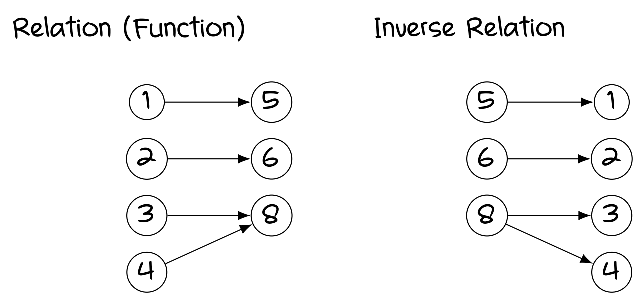
Solution

(a) Swap coordinates: Inverse Relation $= \{(2,3),(1,4),(2,5)\}$.
(b) Swap coordinates: Inverse Relation $= \{(2,1),(4,3),(6,5)\}$.
(c) Swap x and y, then solve for y:

$$x = 3y - 5 \implies x + 5 = 3y \implies y = \frac{x+5}{3}.$$

Important Fact

An inverse relation exists for every function, but the inverse relation is not necessarily a function.



Notice: in the Function, both 3 and 4 map to 8. In the Inverse Relation, 8 maps to both 3 and 4 --- so the inverse relation is not a function.

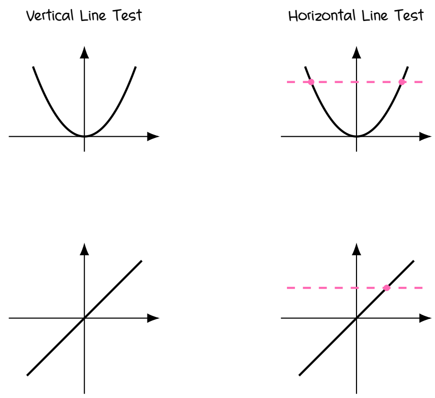
2. One-to-One Functions

One-to-One (1--1) Function

A function is one-to-one if for every a and b in the domain, $a \neq b$ implies $f(a) \neq f(b)$. That is, distinct elements in the domain pair with different elements in the range. Only one-to-one functions have inverses that are also functions.

Horizontal Line Test

If any horizontal line crosses the graph more than once, the function is not one-to-one and does not have an inverse function.



The parabola fails the horizontal line test (not one-to-one, no inverse function); the line passes the horizontal line test (one-to-one, has an inverse function).

Domain and Range of an Inverse

The domain of an inverse, if it exists, is the same as the range of its function. Likewise, the range of the inverse is the same as the domain of the function.

3. Verifying Two Functions are Inverses

Composition Test

Two functions f and g are inverses of each other if

$$(f \circ g)(x) = x \qquad \text{and} \qquad (g \circ f)(x) = x.$$

We denote the inverse of f by f^{-1}.

Example 2: Determine if f and g are inverses of each other.

(a) $f(x) = \dfrac{3}{x+1}\ (x \neq -1)$ and $g(x) = \dfrac{3}{x} - 1\ (x \neq 0)$

(b) $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$

Solution

(a)

$$(f \circ g)(x) = f\left(\frac{3}{x} - 1\right) = \frac{3}{\left(\frac{3}{x} - 1\right) + 1} = \frac{3}{\frac{3}{x}} = 3 \cdot \frac{x}{3} = x. \checkmark$$

$$(g \circ f)(x) = g\left(\frac{3}{x+1}\right) = \frac{3}{\frac{3}{x+1}} - 1 = (x+1) - 1 = x. \checkmark$$

Both compositions equal x, so f and g are inverses of each other.

(b)

$$(f \circ g)(x) = \left(\sqrt[3]{x+1}\right)^3 - 1 = (x+1) - 1 = x. \checkmark$$

$$(g \circ f)(x) = \sqrt[3]{(x^3 - 1) + 1} = \sqrt[3]{x^3} = x. \checkmark$$

Both compositions equal x, so f and g are inverses of each other.

4. Finding the Inverse Function Algebraically

Steps to Find an Inverse Function

1. Rewrite the function in terms of y.
2. Interchange x and y.
3. Solve for y.
4. Rewrite using inverse notation $f^{-1}(x)$.

Example 3: Find the inverse of $f(x) = 4x - 5$.

Solution

$$y = 4x - 5 \implies x = 4y - 5 \implies x + 5 = 4y \implies y = \frac{x+5}{4}.$$

$$\boxed{f^{-1}(x) = \frac{x+5}{4}}$$

Example 4: Find the inverse function, if it exists.

(a) $f(x) = \dfrac{5x+3}{2}$

(b) $g(x) = \dfrac{3x+4}{x-5}\ (x \neq 5)$

(c) $h(x) = 5x^2 + 4\ (x \geq 0)$

Solution

(a)

$$x = \frac{5y+3}{2} \implies 2x = 5y+3 \implies 2x-3 = 5y \implies y = \frac{2x-3}{5}.$$

$$\boxed{f^{-1}(x) = \frac{2x-3}{5}}$$

(b)

$$x = \frac{3y+4}{y-5} \implies x(y-5) = 3y+4 \implies xy - 5x = 3y+4.$$

$$xy - 3y = 5x + 4.$$

$$y(x-3) = 5x+4 \implies y = \frac{5x+4}{x-3}.$$

$$\boxed{g^{-1}(x) = \frac{5x+4}{x-3} \ (x \neq 3)}$$

(c) Since $x \geq 0$, the range of h is $y \geq 4$, so the domain of h^{-1} is $x \geq 4$.

$$x = 5y^2 + 4 \implies x - 4 = 5y^2 \implies y^2 = \frac{x-4}{5} \implies y = \sqrt{\frac{x-4}{5}}$$

(taking the positive root, since $y = x \geq 0$ originally).

$$\boxed{h^{-1}(x) = \sqrt{\frac{x-4}{5}} \ (x \geq 4)}$$

5. Evaluating an Inverse from a Graph or Table

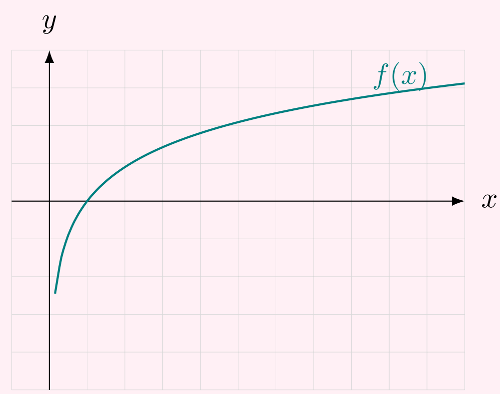
Reading $f^{-1}(a)$ from a Graph

$f^{-1}(a) = b$ means $f(b) = a$. On the graph of f, locate a on the y-axis and read off the corresponding x-value.

Example 5: Use the graph of $f(x)$ below to estimate the following.

(a) $f^{-1}(3)$

(b) $f^{-1}(2)$



Solution

(a) Find where the graph has height $y = 3$: this occurs at approximately $x = 9$. So $f^{-1}(3) \approx 9$.
(b) Find where the graph has height $y = 2$: this occurs at approximately $x = 4$. So $f^{-1}(2) \approx 4$.

Example 6: Use the table below to evaluate the following.

t	0	4	5	10	12	16
$H(t)$	10	22	30	48	25	11

(a) $H^{-1}(48)$

(b) $H^{-1}(11)$

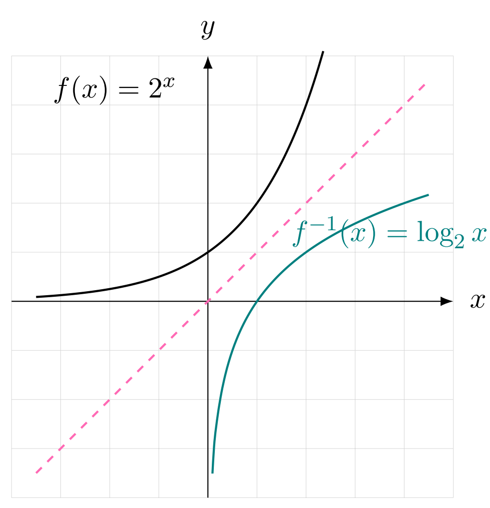
Solution

(a) Find t such that $H(t) = 48$: from the table, $H(10) = 48$. So $H^{-1}(48) = 10$.
(b) Find t such that $H(t) = 11$: from the table, $H(16) = 11$. So $H^{-1}(11) = 16$.

6. Sketching an Inverse from a Graph

Reflection Property

The graphs of a function and its inverse are reflections of one another across the line $y = x$.



Notice that f passes through $(0,1)$, while its reflection f^{-1} passes through $(1,0)$ --- every point (a,b) on f corresponds to a point (b,a) on f^{-1}.

Summary

Key Ideas

- The inverse relation swaps the coordinates of every ordered pair (or interchanges x and y in an equation); every function has an inverse relation, but that relation is not always a function.
- A function is one-to-one if it passes the horizontal line test; only one-to-one functions have an inverse that is also a function.
- The domain of f^{-1} equals the range of f, and the range of f^{-1} equals the domain of f.
- f and g are inverses exactly when $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.
- To find f^{-1} algebraically: write $y = f(x)$, swap x and y, then solve for y.
- The graphs of f and f^{-1} are reflections of each other across the line $y = x$.