Module 4 Lesson 1
Exponential Functions

Objectives

In this lesson, we will learn how to

- identify the initial value and growth/decay factor of an exponential function;
- graph exponential functions and their transformations;
- apply exponential functions to compound interest, population growth, depreciation, and logistic growth;
- determine an exponential function from given data.

1. A Motivating Example

Suppose you are offered a one-month temporary job. Would you choose to be paid \$500 per day, or start at \$0.01 on the first day with your pay doubling each day?

Answer

Choose \$0.01 doubling each day --- by Day 30 it grows to \$5,368,709.12, far more than $500 \times 30 = \$15{,}000$.

Day	Amount	Day	Amount
1	\$0.01	16	\$327.68
2	\$0.02	17	\$655.36
3	\$0.04	18	\$1,310.72
4	\$0.08	19	\$2,621.44
5	\$0.16	20	\$5,242.88
6	\$0.32	21	\$10,485.76
7	\$0.64	22	\$20,971.52
8	\$1.28	23	\$41,973.04
9	\$2.56	24	\$83,886.08
10	\$5.12	25	\$167,772.16
11	\$10.24	26	\$335,544.32
12	\$20.48	27	\$671,088.64
13	\$40.96	28	\$1,342,177.28
14	\$81.92	29	\$2,684,354.56
15	\$163.84	30	\$5,368,709.12

Initial value: $a = 0.01$. Growth factor: $b = 2$.

$$\boxed{f(x) = 0.01(2)^x}$$

represents the salary on day x since the start of the job.

2. General Form

Exponential Function

$$f(x) = ab^x, \qquad \text{where } b > 0 \text{ and } b \neq 1.$$

Here a is the initial value of f, since $f(0) = a$, and b is the growth or decay factor. The function grows if $b > 1$ and decays if $0 < b < 1$.

Example 1: Identify the initial value and growth/decay factor.

(a) $f(x) = 15(3)^x$

(b) $g(x) = 1000\left(\frac{1}{2}\right)^x$

(c) $h(x) = 2^x$

Solution

(a) Initial value: 15. Growth factor: 3.
(b) Initial value: 1000. Decay factor: $\frac{1}{2}$.
(c) $h(x) = 2^x = 1(2)^x$. Initial value: 1. Growth factor: 2.

Example 2: The following are not exponential functions --- give a reason why.

(a) $f(x) = 1^x$

(b) $g(x) = 4x^3$

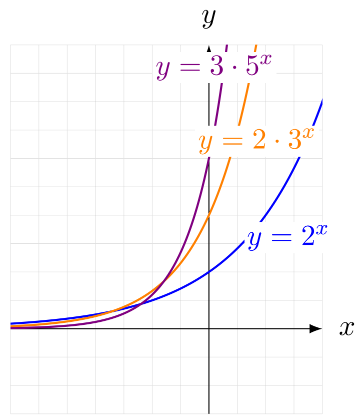
(c) $h(x) = 5\left(-\frac{1}{2}\right)^x$

Solution

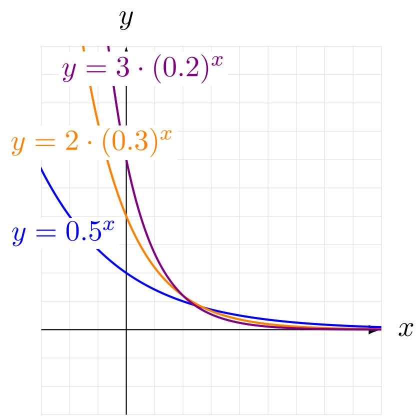
(a) Not exponential, because the base $b = 1$ (the requirement is $b \neq 1$).
(b) Not exponential --- this is a polynomial function (the variable is in the base, not the exponent).
(c) Not exponential, because the base $b = -\frac{1}{2}$ is negative; the base of an exponential function cannot be negative.

3. Graphs of Exponential Functions

$f(x) = ab^x$ with $b > 1$ (growth):



$f(x) = ab^x$ with $0 < b < 1$ (decay):

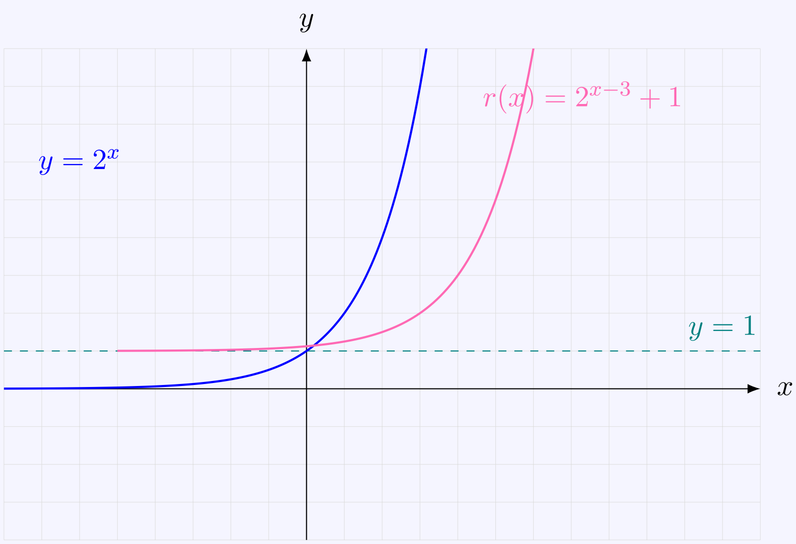


4. Transformations of Exponential Graphs

Example 3: Use the graph of $y = 2^x$ to graph $r(x) = 2^{x-3} + 1$, and determine the domain, range, and horizontal asymptote of $r(x)$.

Solution

The graph of $r(x)$ is $y = 2^x$ shifted 3 units right and 1 unit up.



(a) Domain of $r(x)$: $(-\infty, \infty)$, all real numbers.
(b) Range of $r(x)$: $(1, \infty)$.
(c) Horizontal Asymptote: $y = 1$.

5. Applications of Exponential Functions

Compound Interest

Compound Interest Formulas

If P dollars (principal) is invested at annual rate r (decimal form), the amount A in the account after t years is:

- For n compounding periods per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$
- For continuous compounding: $A = Pe^{rt}$

Example 4: $1000 is invested at 5% interest compounded monthly. How much money is in the account after 3 years?

Solution

$P = 1000,\ r = 0.05,\ n = 12,\ t = 3.$

$$A = P\left(1+\frac{r}{n}\right)^{nt} = 1000\left(1+\frac{0.05}{12}\right)^{(12)(3)} \approx \boxed{\$1161.47}$$

Example 5: $1000 is invested at 5% interest compounded continuously. How much money is in the account after 3 years?

Solution

$P = 1000,\ r = 0.05,\ t = 3.$

$$A = Pe^{rt} = 1000e^{(0.05)(3)} \approx \boxed{\$1161.83}$$

Population Growth

Population Growth Formula

Given an initial population P_0 and annual growth rate r (decimal), the population after t years is

$$P(t) = P_0 e^{rt}.$$

Example 6: A small town's population is modeled by $P(t) = 11500e^{0.025t}$, where t is the number of years since 2020.

(a) Initial population of the town.

(b) Annual growth rate in percentage.

(c) Population of the town in 2030.

Solution

(a) Initial population: $P_0 = 11{,}500$.
(b) Annual growth rate: $r = 0.025 = 2.5\%$.
(c) From 2020 to 2030, $t = 10$ years:

$$P(10) = 11500\, e^{(0.025)(10)} \approx \boxed{14{,}766 \text{ people}}$$

Depreciation

Depreciation Formula

If an item purchased new at price V_0 depreciates at annual rate r (decimal), its value t years after purchase is

$$V = V_0(1 - r)^t.$$

Example 7: An automobile bought new at $27,500 depreciates 6% annually.

(a) Find the formula for the value t years after purchase.

(b) What is the value of the car 4 years after purchase?

(c) How long will it take for the car to be worth only $15,000? Set up the equation.

Solution

(a)

$$V = V_0(1-r)^t = 27500(1-0.06)^t = \boxed{27500(0.94)^t}$$

(b) At $t = 4$:

$$V = 27500(0.94)^4 \approx \boxed{\$21{,}080.22}$$

(c) Set $V = 15000$:

$$\boxed{15000 = 27500(0.94)^t}$$

(This equation is solved using logarithms --- see Lesson 4.4.)

Logistic Growth

Example 8: At a school of $10{,}000$ students, one student returns from Spring break with a contagious flu virus. The spread of the virus is modeled by

$$N = \frac{10000}{1 + 9999e^{-0.6t}},$$

where N is the number of students infected after t days.

(a) How many students are infected after 5 days?

(b) The college will cancel classes if 50% of the students are infected --- when might that occur? Set up the equation.

Solution

(a) Substitute $t = 5$:

$$N = \frac{10000}{1 + 9999e^{-0.6(5)}} \approx \boxed{20 \text{ students}}$$

(b) 50% of $10{,}000$ is 5000, so set $N = 5000$:

$$\boxed{5000 = \frac{10000}{1 + 9999e^{-0.6t}}}$$

(This equation is solved using logarithms --- see Lesson 4.4.)

6. Determining an Exponential Function from Data

Strategy

Start with $f(x) = ab^x$. Substitute two known input/output pairs to get two equations, then divide one equation by the other to eliminate a and solve for b; finally substitute back to find a.

Example 9: An insect population starts at 65 and grows exponentially. After 3 days, the population is 1755. Find the function $P(t)$ that models this growth, where t is the number of days.

Solution

Let $P(t) = ab^t$. We are given $a = 65$ (the initial value) and $P(3) = 1755$:

$$P(3) = 65b^3 \implies 1755 = 65b^3 \implies \frac{1755}{65} = b^3 \implies 27 = b^3.$$

$$3^3 = b^3 \implies b = 3.$$

$$\boxed{P(t) = 65(3)^t}$$

Example 10: A medicine is administered to a patient. The amount present in the bloodstream is 20 mg at $t = 2$ seconds and 160 mg at $t = 5$ seconds. Find the exponential function that models the amount of medicine in the bloodstream as a function of time. What is the initial amount of medicine in the bloodstream?

Solution

Let $A(t) = ab^t$. Substituting $t = 2,\ A = 20$ and $t = 5,\ A = 160$:

$$20 = ab^2 \qquad \text{(Equation 1)}$$

$$160 = ab^5 \qquad \text{(Equation 2)}$$

Divide Equation 2 by Equation 1:

$$\frac{160}{20} = \frac{ab^5}{ab^2} \implies 8 = b^3 \implies 2^3 = b^3 \implies b = 2.$$

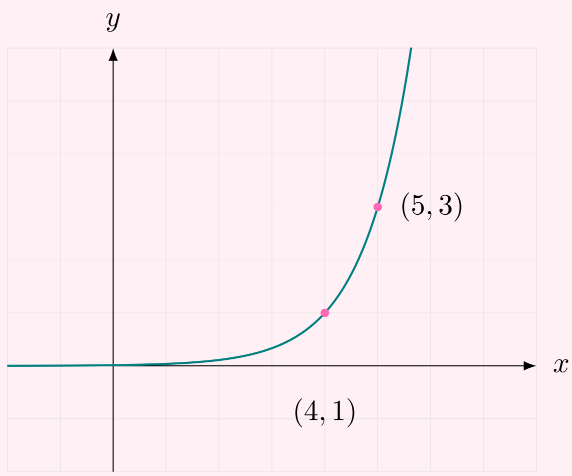
Substitute $b = 2$ into Equation 1:

$$20 = a(2)^2 \implies 20 = 4a \implies a = 5.$$

Thus the initial amount is 5 mg, and

$$\boxed{A(t) = 5(2)^t}$$

Example 11: Determine the exponential function that generates the graph below.



Solution

This is similar to the last problem. Start with $f(x) = ab^x$. Reading two points off the graph: when $x = 4,\ y = 1$, and when $x = 5,\ y = 3$:

$$1 = ab^4 \qquad \text{(Equation 1)}$$

$$3 = ab^5 \qquad \text{(Equation 2)}$$

Divide Equation 2 by Equation 1:

$$\frac{3}{1} = \frac{ab^5}{ab^4} \implies b = 3.$$

Substitute $b = 3$ into Equation 1:

$$1 = a(3)^4 \implies 1 = 81a \implies a = \frac{1}{81}.$$

$$\boxed{f(x) = \frac{1}{81}(3)^x}$$

Summary

Key Ideas

- An exponential function has the form $f(x) = ab^x$ with $b > 0,\ b \neq 1$; $a = f(0)$ is the initial value, and b is the growth ($b > 1$) or decay ($0 < b < 1$) factor.
- Writing $f(x) = ab^{x-h} + k$ shifts the graph of $y = ab^x$: h units horizontally, k units vertically, and the horizontal asymptote becomes $y = k$.
- Applications: compound interest $A = P(1 + r/n)^{nt}$ (or $A = Pe^{rt}$ continuously), population growth $P(t) = P_0e^{rt}$, depreciation $V = V_0(1 - r)^t$, and logistic growth $N = \dfrac{L}{1 + ce^{-kt}}$.
- To determine $f(x) = ab^x$ from two data points, divide the equations to eliminate a and solve for b, then substitute back to find a.