Module 4 Lesson 2
Logarithmic Functions and Graphs

Objectives

In this lesson, we will learn how to

- convert between exponential and logarithmic form;
- evaluate logarithms without a calculator;
- graph logarithmic functions and their transformations;
- apply logarithms to the pH scale and the Richter scale.

1. Motivation and Definition

Let $f(x) = 2^x$. We wish to find the value of x for which $f(x) = 16$, i.e. solve

$$16 = 2^x$$

for x. While we can see that $x = 4$ by inspection, solving for an unknown exponent in more complex problems requires a new operation, the logarithm, where the equation

$$16 = 2^x \qquad \text{is transformed to} \qquad x = \log_2(16).$$

Logarithm Definition

In general,

$$y = a^x \qquad \text{is equivalent to} \qquad x = \log_a(y).$$

Logarithmic functions and exponential functions are inverses of each other.

Example 1: Convert each logarithmic equation to an exponential equation.

(a) $\log_2 8 = 3$

(b) $\log_5 25 = 2$

(c) If $\log_2(y) = 5$, what is the value of y?

Solution

(a) $\log_2 8 = 3$ is equivalent to $2^3 = 8$.
(b) $\log_5 25 = 2$ is equivalent to $5^2 = 25$.
(c) $\log_2(y) = 5$ means $y = 2^5 = \boxed{32}$.

Natural and Common Logarithms

Natural Log: $\log_e x \Leftrightarrow \ln x$

Common Log: $\log_{10} x \Leftrightarrow \log x$

Example 2: Convert each exponential equation to a logarithmic equation.

(a) $3^4 = 81$

(b) $10^6 = 1{,}000{,}000$

(c) $e^2 \approx 7.3891$

Solution

(a) $3^4 = 81$ is equivalent to $\log_3(81) = 4$.
(b) $10^6 = 1{,}000{,}000$ is equivalent to $\log(1{,}000{,}000) = 6$.
(c) $e^2 \approx 7.3891$ is equivalent to $\ln(7.3891) \approx 2$.

Example 3: Find each of the following (do not use a calculator).

(a) $\log_4 16$

(b) $\log_9 3$

(c) $\log_2 32$

(d) $\log_5 \left(\frac{1}{25}\right)$

(e) $\log(1000)$

(f) $\ln(e^4)$

(g) $\log_2(-4)$

Solution

(a) $4^x = 16 = 4^2 \implies \log_4 16 = 2$.

(b) $9^x = 3 \implies (3^2)^x = 3^1 \implies 2x = 1 \implies \log_9 3 = \frac{1}{2}$.

(c) $2^x = 32 = 2^5 \implies \log_2 32 = 5$.

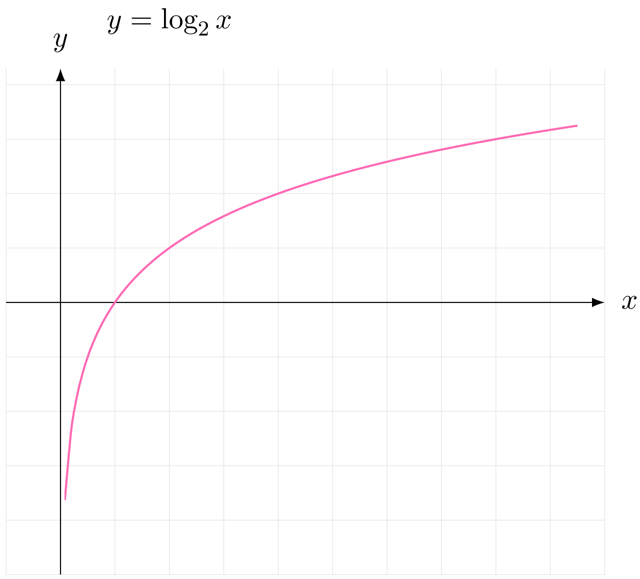
(d) $5^x = \frac{1}{25} = 5^{-2} \implies \log_5 \left(\frac{1}{25}\right) = -2$.

(e) $\log(1000) = \log(10^3) = 3$.

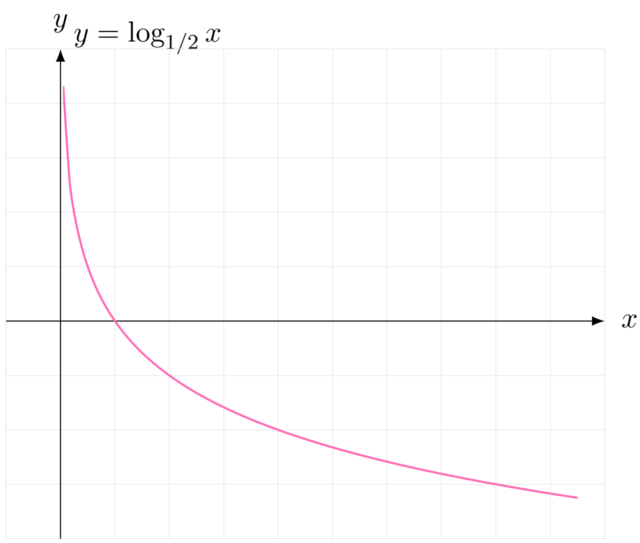
(f) $\ln(e^4) = 4$.

(g) $\log_2(-4)$ is undefined --- a logarithm of a negative number does not exist (there is no real power of 2 that equals -4).

2. Graphing Logarithmic Functions



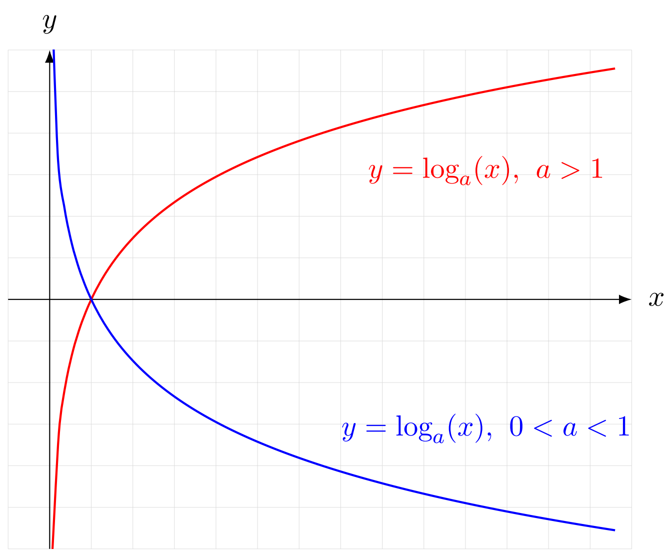
Domain: $(0, \infty)$. Range: $(-\infty, \infty)$. Vertical Asymptote: $x = 0$.



Domain: $(0, \infty)$. Range: $(-\infty, \infty)$. Vertical Asymptote: $x = 0$.

Graph of $y = \log_a(x)$ in General

- Domain: $(0, \infty)$. Range: $(-\infty, \infty)$.
- Vertical Asymptote: $x = 0$.
- x-intercept: $(1, 0)$, since $\log_a(1) = 0$ for any base a.
- y-intercept: none --- $x = 0$ is not in the domain.

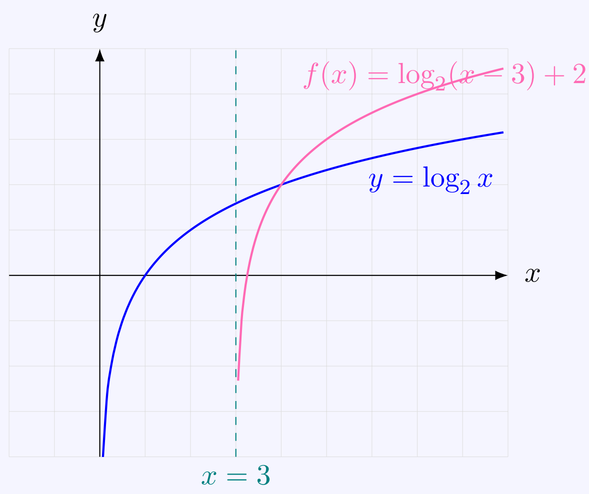


3. Transformations of Logarithmic Graphs

Example 4: Graph $f(x) = \log_2(x - 3) + 2$. Determine the domain, range, and vertical asymptote.

Solution

This is $y = \log_2(x)$ shifted 3 units right and 2 units up.



Domain: $x - 3 > 0 \implies x > 3$, i.e. $(3, \infty)$.
Range: $(-\infty, \infty)$ --- a vertical shift never restricts the range of a log function.
Vertical Asymptote: $x = 3$.

Example 5: Determine the domain and vertical asymptote for the graph of $g(x) = \log(6-x) - 3$.

Solution

This is $y = \log(x)$ reflected about the y-axis, shifted right 6, and down 3.
Domain: we need $6 - x > 0 \implies x < 6$, i.e. $\boxed{(-\infty, 6)}$.
Vertical Asymptote: $6 - x = 0 \implies \boxed{x = 6}$.

4. Applications

The pH Scale

pH Formula

In chemistry, pH is defined by

$$pH = -\log[H^+],$$

where $[H^+]$ is the concentration of the hydrogen ion (in moles per liter).

Example 6: Find the pH of the following.

(a) Lemon Juice, $H^+ \approx 1.3 \times 10^{-5}$

(b) Apples, $H^+ \approx 2.7 \times 10^{-6}$

Solution

(a)

$$pH = -\log(1.3 \times 10^{-5}) \approx \boxed{4.89}$$

(b)

$$pH = -\log(2.7 \times 10^{-6}) \approx \boxed{5.57}$$

Example 7: Find the H^+ concentration of the following.

(a) Soda Ash, with $pH = 10.3$

(b) Bottled Water, with $pH = 7$

Solution

Solve $pH = -\log[H^+]$ for $[H^+]$: $[H^+] = 10^{-pH}$.

(a)

$$[H^+] = 10^{-10.3} \approx \boxed{5.01 \times 10^{-11} \text{ moles/liter}}$$

(b)

$$[H^+] = 10^{-7} \text{ moles/liter}$$

The Richter Scale

Richter Scale Formula

For an earthquake of intensity I, its magnitude R on the Richter scale is

$$R = \log \frac{I}{I_0},$$

where I_0 is the reference intensity --- the smallest earth movement detectable on a seismograph. Each unit rise on the Richter scale corresponds to a 10 times increase in intensity.

Example 8: Find the magnitude of an earthquake whose intensity is

(a) $50{,}000\, I_0$

(b) $500{,}000\, I_0$

Solution

(a)

$$R = \log\left(\frac{50000\, I_0}{I_0}\right) = \log(50000) \approx \boxed{4.70}$$

(b)

$$R = \log\left(\frac{500000\, I_0}{I_0}\right) = \log(500000) \approx \boxed{5.70}$$

Example 9: How many times more intense is an earthquake with a magnitude of 6.7 than one with a magnitude of 4.7 on the Richter Scale?

Solution

Each unit on the Richter scale is a factor of 10 in intensity, so

$$10^{6.7-4.7} = 10^2 = \boxed{100 \text{ times more intense}}$$

Summary

Key Ideas

- $y = a^x$ is equivalent to $x = \log_a(y)$; logarithmic and exponential functions are inverses of each other.
- Natural log: $\log_e x = \ln x$. Common log: $\log_{10} x = \log x$.
- $\log_a(x)$ is only defined for $x > 0$ --- the logarithm of zero or a negative number does not exist.
- The graph of $y = \log_a(x)$ has domain $(0, \infty)$, range $(-\infty, \infty)$, vertical asymptote $x = 0$, and x-intercept $(1, 0)$ (no y-intercept).
- Writing $f(x) = \log_a(x-h)+k$ shifts the graph h units horizontally and k units vertically, moving the vertical asymptote to $x = h$; the range always stays $(-\infty, \infty)$.
- Applications: pH $= -\log[H^+]$, and the Richter scale $R = \log(I/I_0)$, where each whole-number increase means $10\times$ the intensity.