# Module 4 Lesson 3
# Properties of Logarithms

## Objectives

**In this lesson, we will learn how to**

- apply the basic properties of logarithms;
- expand a logarithm as a sum or difference;
- combine logarithms into a single logarithm;
- use the change-of-base formula to evaluate logarithms.

## 1. Basic Properties of Logarithms

**Basic Properties**

From the definition of a logarithmic function, we have:

(a) $\log_a(1) = 0$, because $a^0 = 1$.

(b) $\log_a(a) = 1$, because $a^1 = a$.

(c) $\log_a(a^p) = p$, because raising $a$ to the $p$ gives $a^p$ directly.

(d) $a^{\log_a(x)} = x$, because $\log_a(x)$ is exactly the exponent that, applied to $a$, produces $x$.

## Additional Properties

(e) $\log_a(xy) = \log_a(x) + \log_a(y)$ ( Product Rule )

(f) $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$ ( Quotient Rule )

(g) $\log_a(x^p) = p\log_a(x)$ ( Power Rule )

Example 1: Apply the properties of logarithms to compute:

(a) $\log_5(5^4)$

(b) $\log_6(1)$

(c) $4^{\log_4(12)}$

(d) $\log_3(3)$

(e) $\log_4(16 \cdot 4)$

## Solution

(a) $\log_5(5^4) = 4$ (property c).
(b) $\log_6(1) = 0$ (property a).
(c) $4^{\log_4(12)} = 12$ (property d).
(d) $\log_3(3) = 1$ (property b).
(e) $\log_4(16 \cdot 4) = \log_4(64) = \log_4(4^3) = 3$.

## 2. Expanding Logarithms

**Example 2:** Expand the following as a sum or difference of logarithms.

(a) $\log(100P)$

(b) $\ln\left(\sqrt[5]{6}\right)$

(c) $\log_c\left(\dfrac{14}{23}\right)$

(d) $\log_3\left(\dfrac{x^2y^5}{m^3n^4}\right)$

**Solution**

(a) $\log(100P) = \log(100) + \log(P) = 2 + \log(P)$.

(b) $\ln\left(\sqrt[5]{6}\right) = \ln\left(6^{1/5}\right) = \dfrac{1}{5}\ln(6)$.

(c) $\log_c\left(\dfrac{14}{23}\right) = \log_c(14) - \log_c(23)$.

(d) $\log_3\left(\dfrac{x^2y^5}{m^3n^4}\right) = 2\log_3(x) + 5\log_3(y) - 3\log_3(m) - 4\log_3(n)$.

**Example 3:** Expand the following as a sum or difference of logarithms.

(a) $\ln\sqrt{\dfrac{x^3y^4}{z^6}}$

(b) $\ln\sqrt{\dfrac{x^2+6}{e^3}}$

Solution

(a)

$$\ln\sqrt{\frac{x^3y^4}{z^6}} = \frac{1}{2}\ln\left(\frac{x^3y^4}{z^6}\right) = \frac{1}{2}\left[3\ln x + 4\ln y - 6\ln z\right]$$

$$= \frac{3}{2}\ln x + 2\ln y - 3\ln z.$$

(b) Note $x^2+6$ does not split further (it is a sum, not a product), but $\ln(e^3) = 3$:

$$\ln\sqrt{\frac{x^2+6}{e^3}} = \frac{1}{2}\left[\ln(x^2+6) - \ln(e^3)\right] = \frac{1}{2}\ln(x^2+6) - \frac{3}{2}.$$

## 3. Combining Logarithms

Example 4: Express the following as a single logarithm.

(a) $\log_a 14 + \log_a 3$

(b) $2\ln x + 3(\ln y - \ln z)$

(c) $\frac{2}{3}\left[\ln(z^2-9) - \ln(z+3)\right] + \ln(z-3)$

(d) $\frac{1}{3}\ln(x^2+1) - \frac{1}{3}\ln(x-3)$

Solution

(a) $\log_a 14 + \log_a 3 = \log_a(14 \cdot 3) = \log_a(42)$.

(b) $2 \ln x + 3(\ln y - \ln z) = \ln(x^2) + \ln\left(\frac{y^3}{z^3}\right) = \ln\left(\frac{x^2 y^3}{z^3}\right)$.

(c) Since $z^2 - 9 = (z-3)(z+3)$, we have $\frac{z^2 - 9}{z+3} = z - 3$, so

$$\begin{aligned}\frac{2}{3}\left[\ln(z^2 - 9) - \ln(z+3)\right] + \ln(z-3) &= \frac{2}{3}\ln(z-3) + \ln(z-3) \\ &= \frac{5}{3}\ln(z-3) = \ln\left[(z-3)^{5/3}\right].\end{aligned}$$

(d)

$$\frac{1}{3}\ln(x^2+1) - \frac{1}{3}\ln(x-3) = \frac{1}{3}\ln\left(\frac{x^2+1}{x-3}\right) = \ln\left[\left(\frac{x^2+1}{x-3}\right)^{1/3}\right].$$

## 4. Estimating Logarithms

Example 5: Let $\log_a 2 \approx .43$, $\log_a 3 \approx .68$, and $\log_a 5 \approx 1.55$. Estimate the following.

(a) $\log_a 6$

(b) $\log_a\left(\frac{2}{3}\right)$

(c) $\log_a 20$

**Solution**

(a) $\log_a 6 = \log_a(2 \cdot 3) = \log_a 2 + \log_a 3 \approx .43 + .68 = \boxed{1.11}$.

(b) $\log_a \left(\dfrac{2}{3}\right) = \log_a 2 - \log_a 3 \approx .43 - .68 = \boxed{-.25}$.

(c) $\log_a 20 = \log_a(4 \cdot 5) = \log_a(2^2 \cdot 5) = 2\log_a 2 + \log_a 5 \approx 2(.43) + 1.55 = \boxed{2.41}$.

## 5. Change of Base Formula

**Change of Base Formula**

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

Examples:

$$\log_5(x) = \frac{\log_{10}(x)}{\log_{10}(5)} \qquad \log_5(x) = \frac{\ln(x)}{\ln(5)}$$

**Example 6:** Compute $\log_5(33)$ by changing to base $10$.

**Solution**

$$\log_5(33) = \frac{\log_{10}(33)}{\log_{10}(5)} \approx \frac{1.5185}{0.6990} \approx \boxed{2.17}$$

## Summary

### Key Ideas

- Basic properties: $\log_a(1) = 0$, $\log_a(a) = 1$, $\log_a(a^p) = p$, and $a^{\log_a(x)} = x$.
- Product Rule: $\log_a(xy) = \log_a(x) + \log_a(y)$. Quotient Rule: $\log_a(x/y) = \log_a(x) - \log_a(y)$. Power Rule: $\log_a(x^p) = p\log_a(x)$.
- These rules only split products, quotients, and powers inside a logarithm --- a sum or difference inside a logarithm, like $\log(x^2+6)$, cannot be split further.
- Change of Base Formula: $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$, useful for evaluating a logarithm in a base your calculator doesn't have (typically using base $10$ or base $e$).