Module 4 Lesson 4A
Logarithmic Equations and Applications

Objectives

In this lesson, we will learn how to

- solve logarithmic equations using the equivalence property;
- combine logarithms first when an equation has multiple log terms;
- check solutions against the domain of the original logarithms;
- apply logarithmic equations to the pH scale.

1. Equivalence Property

Equivalence Property of Logarithmic Equations

Let a, x, y be positive real numbers with $a \neq 1$. If

$$\log_a x = \log_a y \qquad \text{then} \qquad x = y.$$

A Reminder

Since $\log_a(x)$ is only defined for $x > 0$, always check that a candidate solution keeps every logarithm's argument positive in the original equation. A solution that fails this check is extraneous and must be discarded.

Example 1: Solve.

(a) $\log(x^2) = \log(16)$

(b) $\log_5(8 - 7x) = 3$

Solution

(a) By the equivalence property, $x^2 = 16 \implies x = \pm 4$. Both keep $x^2 = 16 > 0$, so both are valid.

$$\boxed{x = 4, \ \ x = -4}$$

(b) Rewrite in exponential form: $8 - 7x = 5^3 = 125$.

$$-7x = 117 \implies x = -\frac{117}{7}$$

Check: $8 - 7x = 125 > 0$. ✓

$$\boxed{x = -\frac{117}{7}}$$

2. Combining Logarithms Before Solving

When an equation has multiple logarithmic terms, first combine them into a single logarithm using the product/quotient/power rules, then apply the equivalence property.

Example 2: Solve $\ln x - \ln(x - 4) = \ln 3$.

Solution

$$\ln\left(\frac{x}{x-4}\right) = \ln 3 \implies \frac{x}{x-4} = 3 \implies x = 3(x-4) = 3x - 12.$$

$$-2x = -12 \implies x = 6.$$

Check: $x = 6 > 0$ and $x - 4 = 2 > 0$. ✓

$$\boxed{x = 6}$$

Example 3: Solve $\log_2(x-4) = \log_2(2) - \log_2(x-3)$.

Solution

Combine the right side: $\log_2(2) - \log_2(x-3) = \log_2\left(\frac{2}{x-3}\right)$.

$$\log_2(x-4) = \log_2\left(\frac{2}{x-3}\right) \implies x-4 = \frac{2}{x-3} \implies (x-4)(x-3) = 2$$

$$x^2 - 7x + 12 = 2 \implies x^2 - 7x + 10 = 0 \implies (x-5)(x-2) = 0.$$

$$x = 5 \text{ or } x = 2.$$

Check $x = 5$: $x - 4 = 1 > 0$ and $x - 3 = 2 > 0$. ✓
Check $x = 2$: $x - 4 = -2 < 0$. Fails --- extraneous.

$$\boxed{x = 5}$$

Example 4: Solve $2\ln x - \ln 5 = \ln(x+10)$.

Solution

$$\ln(x^2) - \ln 5 = \ln(x+10) \implies \ln\left(\frac{x^2}{5}\right) = \ln(x+10) \implies \frac{x^2}{5} = x+10$$

$$x^2 = 5x + 50 \implies x^2 - 5x - 50 = 0 \implies (x-10)(x+5) = 0.$$

$$x = 10 \text{ or } x = -5.$$

Check $x = 10$: $x > 0$ and $x + 10 = 20 > 0$. ✓
Check $x = -5$: $\ln x$ requires $x > 0$, but $x = -5 < 0$. Fails --- extraneous.

$$\boxed{x = 10}$$

Example 5: Solve $\log_3(\log_2 x) = 0$.

Solution

Rewrite in exponential form (base 3): $\log_2 x = 3^0 = 1$.
Rewrite again (base 2): $x = 2^1 = 2$.

$$\boxed{x = 2}$$

3. Application: The pH Scale

pH Formula

The acidity of a solution is measured using the pH scale, defined by

$$pH = -\log[H^+],$$

where $[H^+]$ is the concentration of the hydrogen ion in moles per liter.

Example 6: A certain solution has pH of 3.5. What is the hydrogen ion concentration $[H^+]$ in the solution?

Solution

$$3.5 = -\log[H^+] \implies \log[H^+] = -3.5 \implies [H^+] = 10^{-3.5}$$

$$\boxed{[H^+] \approx 3.16 \times 10^{-4} \text{ moles/liter}}$$

Summary

Key Ideas

- Equivalence Property: if $\log_a x = \log_a y$ (same base), then $x = y$.
- If an equation has a single logarithm equal to a number, rewrite it in exponential form: $\log_a(x) = p \implies x = a^p$.
- If an equation has multiple logarithmic terms, first combine them into a single logarithm with the product/quotient/power rules, then apply the equivalence property.
- Always check every solution against the domain of each original logarithm (argument > 0) --- solving often introduces extraneous roots that must be discarded.
- Application: $pH = -\log[H^+]$, so $[H^+] = 10^{-pH}$.