# Module 4 Lesson 4
# Exponential Equations and Applications

## Objectives

**In this lesson, we will learn how to**

- solve exponential equations by matching bases;
- solve exponential equations using logarithms;
- apply exponential equations to compound interest problems.

## 1. Equivalence Property

**Equivalence Property of Exponential Equations**

Let $a, x, y$ be real numbers with $a > 0$ and $a \neq 1$. Then

$$a^x = a^y \qquad \text{implies that} \qquad x = y.$$

Example: $5^x = 25 \implies 5^x = 5^2 \implies x = 2$.

Example 1: Solve the following exponential equations.

(a) $3^x = 27$

(b) $4^{2x-3} = 32$

(c) $8^{x+2} = 16^{x+1}$

Solution

(a) $3^x = 27 = 3^3 \implies \boxed{x = 3}$.
(b) Write both sides with base $2$: $4 = 2^2$, $32 = 2^5$.

$$2^{2(2x-3)} = 2^5 \implies 4x - 6 = 5 \implies 4x = 11 \implies \boxed{x = \frac{11}{4}}$$

(c) Write both sides with base $2$: $8 = 2^3$, $16 = 2^4$.

$$2^{3(x+2)} = 2^{4(x+1)} \implies 3x + 6 = 4x + 4 \implies \boxed{x = 2}$$

## 2. Solving Exponential Equations Using Logarithms

Some problems cannot be solved by setting the powers equal to each other (since the two sides cannot be written with matching bases). These are solved by applying a logarithm to both sides of the equation.

Example 2: Solve the following equations.

(a) $e^x = 1200$

(b) $5^{2x-3} = 17$

(c) $319 = 11^x + 5$

Solution

(a) Apply $\ln$ to both sides:

$$\ln(e^x) = \ln(1200) \implies x = \ln(1200) \approx \boxed{7.09}$$

(b) Apply $\ln$ to both sides:

$$(2x-3)\ln 5 = \ln 17 \implies 2x - 3 = \frac{\ln 17}{\ln 5} \approx 1.7604$$

$$2x \approx 4.7604 \implies \boxed{x \approx 2.38}$$

(c) Isolate the exponential first:

$$319 - 5 = 11^x \implies 314 = 11^x \implies x = \frac{\ln 314}{\ln 11} \approx \boxed{2.40}$$

Example 3: Solve the following equations.

(a) $60 = 80e^{-0.4x}$

(b) $3^{x-2} = 5^{x+1}$

Solution

(a) Isolate the exponential, then apply $\ln$:

$$\frac{60}{80} = e^{-0.4x} \implies 0.75 = e^{-0.4x} \implies \ln(0.75) = -0.4x$$

$$x = \frac{\ln(0.75)}{-0.4} \approx \boxed{0.72}$$

(b) Apply $\ln$ to both sides:

$$(x-2)\ln 3 = (x+1)\ln 5$$

$$x\ln 3 - 2\ln 3 = x\ln 5 + \ln 5$$

$$x(\ln 3 - \ln 5) = \ln 5 + 2\ln 3$$

$$x = \frac{\ln 5 + 2\ln 3}{\ln 3 - \ln 5} \approx \frac{1.6094 + 2.1972}{1.0986 - 1.6094} = \frac{3.8066}{-0.5108} \approx \boxed{-7.45}$$

## 3. Application: Compound Interest

Example 4: If \$5000 is put into an interest-bearing account at an interest rate of $2.2\%$ compounded monthly, find the time required for the account to earn \$1000. Round to the nearest month.

### Solution

Earning \$1000 means the account grows from \$5000 to \$6000, so $A = 6000$, $P = 5000$, $r = 0.022$, $n = 12$.

$$6000 = 5000\left(1 + \frac{0.022}{12}\right)^{12t}$$

$$1.2 = (1.0018333)^{12t}$$

Apply $\ln$ to both sides:

$$\ln(1.2) = 12t\ \ln(1.0018333)$$

$$12t = \frac{\ln(1.2)}{\ln(1.0018333)} \approx \frac{0.18232}{0.0018317} \approx 99.55$$

Rounding to the nearest month, $12t \approx \boxed{100 \text{ months}}$ (about $8$ years and $4$ months).

## Summary

### Key Ideas

- Equivalence Property: if $a^x = a^y$ (same base), then $x = y$. Rewrite both sides with a common base whenever possible.
- When the two sides cannot share a common base, apply $\ln$ (or $\log$) to both sides of the equation, then use the power rule $\ln(b^x) = x\ln(b)$ to bring the variable down out of the exponent.
- Always isolate the exponential expression first before applying a logarithm (e.g. subtract or divide off any extra terms/factors).
- When variable exponents appear on both sides with different bases (like $3^{x-2} = 5^{x+1}$), take $\ln$ of both sides, expand with the product rule, then collect all $x$-terms on one side to solve.