Chapter 4 Probability
Addition Rule and Multiplication Rule (4.2)

Objectives

In this lesson, we will learn how to

- recognize compound events;
- use the addition rule to find $P(A \text{ or } B)$;
- identify disjoint events;
- use complements to find probabilities;
- use the multiplication rule to find $P(A \text{ and } B)$;
- distinguish between independent and dependent events;
- apply the 5% guideline for cumbersome calculations.

1. Key Ideas

In probability, the words or and and are very important.

Two Main Rules

- The word or is connected with addition.
- The word and is connected with multiplication.

In this lesson, we use these rules to find probabilities of compound events.

Compound Event

A compound event is any event that combines two or more simple events.

For example, when rolling one die, the event ``getting an even number'' is a compound event because it contains the simple events

$$2,\ 4,\ 6.$$

2. Addition Rule: The Word "Or"

The notation

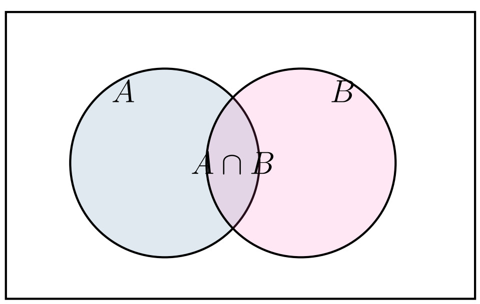
$$P(A \text{ or } B)$$

means the probability that event A occurs or event B occurs, or they both occur.

Formal Addition Rule

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

The subtraction avoids counting the overlap twice.



$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Intuitive Addition Rule

To find $P(A \text{ or } B)$, add the number of ways event A can occur and the number of ways event B can occur, but count each outcome only once.

Example 1: A card is randomly selected from a standard deck of 52 cards. Find the probability of selecting a heart or a king.

Solution

Let

$$A = \text{selecting a heart}, \qquad B = \text{selecting a king}.$$

There are 13 hearts, so

$$P(A) = \frac{13}{52}.$$

There are 4 kings, so

$$P(B) = \frac{4}{52}.$$

The overlap is the king of hearts, so

$$P(A \text{ and } B) = \frac{1}{52}.$$

Using the addition rule,

$$P(A \text{ or } B) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13} \approx 0.308.$$

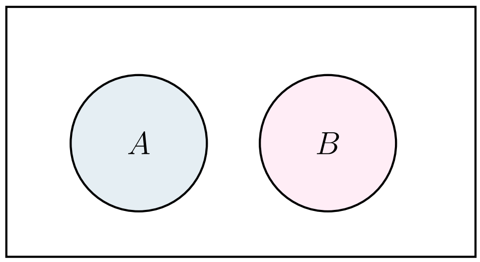
So the probability of selecting a heart or a king is approximately

$$0.308.$$

3. Disjoint Events

Disjoint Events

Events A and B are disjoint, or mutually exclusive, if they cannot occur at the same time.



Disjoint events do not overlap.

If events A and B are disjoint, then

$$P(A \text{ and } B) = 0.$$

So the addition rule becomes simpler:

Addition Rule for Disjoint Events

If A and B are disjoint, then

$$P(A \text{ or } B) = P(A) + P(B).$$

Example 2: Decide whether the events are disjoint.

(a) Randomly selecting a person who is male and randomly selecting a person who is female.

(b) Randomly selecting a student taking a statistics course and randomly selecting a student who is female.

Solution

(a) These events are disjoint, because one selected person cannot be both male and female in this classification.

(b) These events are not disjoint, because a selected person can be both taking a statistics course and female.

Example 3: A single die is rolled. Find the probability of getting a 1 or a 6.

Solution

Let

$$A = \text{getting a 1}, \qquad B = \text{getting a 6}.$$

These events are disjoint, because one roll cannot be both 1 and 6. Therefore,

$$P(A \text{ or } B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}.$$

So the probability is

$$\frac{1}{3} \approx 0.333.$$

4. Complementary Events

The complement of event A is the event that A does not occur.
We write the complement as

$$\overline{A}.$$

Rule of Complementary Events

$$P(A) + P(\overline{A}) = 1.$$

Equivalently,

$$P(\overline{A}) = 1 - P(A).$$

Example 4: Smartphone Homes: Based on survey results, the probability of randomly selecting a U.S. household with a smartphone is

$$P(\text{smartphone}) = 0.87.$$

Find the probability of selecting a household that does not have a smartphone.

Solution

Use the complement rule:

$$P(\text{not smartphone}) = 1 - P(\text{smartphone}).$$

Substitute:

$$P(\text{not smartphone}) = 1 - 0.87 = 0.13.$$

So the probability is

$$0.13.$$

5. Multiplication Rule: The Word "And"

The notation

$$P(A \text{ and } B)$$

means the probability that event A occurs and event B occurs.

Conditional Probability Notation

$$P(B \mid A)$$

means the probability of event B occurring after assuming that event A has already occurred.

Formal Multiplication Rule

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A).$$

Intuitive Multiplication Rule

To find the probability that event A occurs and then event B occurs, multiply the probability of A by the probability of B, after taking into account that A has already occurred.

6. Independent and Dependent Events

Independent Events

Two events are independent if the occurrence of one event does not affect the probability of the other event.

Dependent Events

Two events are dependent if the occurrence of one event does affect the probability of the other event.

Sampling Connection

- Sampling with replacement: selections are independent.
- Sampling without replacement: selections are dependent.

Example 5: Drug Screening: Among 50 test results from subjects who use drugs, 45 are positive and 5 are negative.

Result	Count
Positive	45
Negative	5
Total	50

(a) If 2 subjects are randomly selected with replacement, find the probability that the first selected person had a positive result and the second selected person had a negative result.

(b) Repeat part (a), but assume the two subjects are selected without replacement.

Solution

(a) With replacement
The first probability is

$$P(\text{positive first}) = \frac{45}{50}.$$

Because the first result is replaced, the second probability is still based on 50 results:

$$P(\text{negative second}) = \frac{5}{50}.$$

Therefore,

$$P(\text{positive first and negative second}) = \frac{45}{50} \cdot \frac{5}{50} = 0.09.$$

(b) Without replacement
The first probability is still

$$P(\text{positive first}) = \frac{45}{50}.$$

After one positive result is selected and not replaced, 49 results remain, with 5 negative results. So

$$P(\text{negative second} \mid \text{positive first}) = \frac{5}{49}.$$

Therefore,

$$P(\text{positive first and negative second}) = \frac{45}{50} \cdot \frac{5}{49} \approx 0.0918.$$

7. The 5% Guideline

Sometimes we sample without replacement, but the population is so large that the probabilities barely change.

5% Guideline for Cumbersome Calculations

When sampling without replacement and the sample size is no more than 5% of the population size, we may treat the selections as independent, even though they are actually dependent.

Example 6: Drug Screening and the 5% Guideline: In a recent year, there were 130,639,273 full-time employees in the United States. Suppose the probability that one randomly selected employee tests positive for illegal drug use is 0.042.
If three employees are randomly selected without replacement, find the probability that all three test positive.

Solution

The sample size is 3, which is much less than 5% of 130,639,273. So we can treat the selections as independent.
Therefore,

$$P(\text{all 3 test positive}) = (0.042)(0.042)(0.042).$$

$$= 0.0000741.$$

So the probability that all three selected employees test positive is

$$0.0000741.$$

8. Redundancy and Reliability

Redundancy means using extra components so that a system is more reliable.

For example, if a computer has two hard drives and only one needs to work, then the computer is less likely to fail completely.

Example 7: Hard Drive Reliability: A hard drive has a failure rate of 2.89% in a year.

(a) If a computer has one such hard drive, what is the probability that it works for a year?

(b) If a computer has two independent such hard drives, what is the probability that at least one works for a year?

Solution

The probability of failure is

$$0.0289.$$

(a) One hard drive
The probability that the hard drive works is

$$1 - 0.0289 = 0.9711 \approx 0.971.$$

(b) Two hard drives
The computer fails only if both hard drives fail. Since the hard drives are independent,

$$P(\text{both fail}) = (0.0289)(0.0289) = 0.00083521.$$

So the probability that at least one works is

$$1 - P(\text{both fail}) = 1 - 0.00083521 = 0.99916479.$$

Rounded to three significant digits,

$$0.999.$$

9. Summary of the Two Rules

Addition Rule

The word or suggests addition.

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

If A and B are disjoint, then

$$P(A \text{ or } B) = P(A) + P(B).$$

Multiplication Rule

The word and suggests multiplication.

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A).$$

If A and B are independent, then

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

Key Reminders

- Use addition for or problems.
- Use multiplication for and problems.
- Avoid double-counting when using the addition rule.
- Adjust the second probability when selections are made without replacement.
- Use complements when it is easier to find the probability of the opposite event.