Chapter 4 Probability
Counting (4.4)

Objectives

In this lesson, we will learn how to

- use the multiplication counting rule;
- use factorials to count arrangements;
- distinguish between permutations and combinations;
- count permutations when order matters;
- count combinations when order does not matter;
- apply counting rules to probability problems.

1. Why Counting Matters in Probability

Many probability problems require us to count the number of possible outcomes.

Key Idea

If outcomes are equally likely, then

$$P(A) = \frac{\text{number of outcomes in event } A}{\text{total number of outcomes}}.$$

So, good counting helps us find probabilities correctly.

This section introduces several common counting methods. Not every counting problem fits one of these methods exactly, but these rules give a strong foundation for many real applications.

2. Multiplication Counting Rule

Multiplication Counting Rule

For a sequence of events, if

- the first event can occur in n_1 ways,
- the second event can occur in n_2 ways,
- the third event can occur in n_3 ways,

then the total number of outcomes is

$$n_1 \cdot n_2 \cdot n_3 \cdots .$$

Example 1: Hacker Guessing a Passcode: A password has five hidden characters. Suppose each character can be selected from 92 possible keyboard characters.

(a) How many different passwords are possible?

(b) If all passwords are equally likely, what is the probability of guessing the correct password on the first attempt?

Solution

For each of the five positions, there are 92 choices.

$$92 \cdot 92 \cdot 92 \cdot 92 \cdot 92 = 92^5.$$

$$92^5 = 6{,}590{,}815{,}232.$$

So there are

$$6{,}590{,}815{,}232$$

different possible passwords.

Since exactly one password is correct,

$$P(\text{correct on first attempt}) = \frac{1}{6{,}590{,}815{,}232} \approx 0.000000000152.$$

3. Factorial Rule

Factorial Notation

The factorial symbol ! means the product of decreasing positive whole numbers.

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1.$$

By special definition,

$$0! = 1.$$

Examples:

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120,$$

$$3! = 3 \cdot 2 \cdot 1 = 6.$$

Factorial Rule

The number of different arrangements of n different items, when all n items are selected, is

$$n!.$$

This rule is used when order matters.

Example 2: Scrambling Letters: Consider the word steam.

(a) How many different ways can the letters of steam be arranged?

(b) If the letters are arranged randomly, what is the probability that the letters will be in alphabetical order?

(c) How many different real English words can be formed with the letters in steam?

Solution

The word steam has five different letters.

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120.$$

So the letters can be arranged in 120 different ways.
There is only one alphabetical arrangement:

$$aemst.$$

Therefore,

$$P(\text{alphabetical order}) = \frac{1}{120}.$$

For part (c), the counting rule gives arrangements of letters, not necessarily real English words. To answer part (c), we would need to know which of the 120 arrangements are actual words.

4. Permutations and Combinations

Permutation

A permutation is an arrangement in which different orders are counted separately.
Order matters.

For example, the letter arrangements

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba$$

are counted as six different permutations.

Combination

A combination is a selection in which different orders are counted as the same.
Order does not matter.

For example, if a, b, c are selected as a committee, then

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba$$

all represent the same combination.

Memory Trick

- Permutations Position: with permutations, position matters.
- Combinations Committee: with a committee, order does not matter.

5. Permutations Rule

Permutations Rule

When n different items are available and r of them are selected without replacement, the number of different permutations is

$$_nP_r = \frac{n!}{(n-r)!}.$$

Use this rule when order matters.

Example 3: Trifecta Bet: In a horse race, a trifecta bet is won by correctly selecting the horses that finish first, second, and third in the correct order. Suppose there are 20 horses.

(a) How many different trifecta bets are possible?

(b) If one arrangement is randomly selected, what is the probability of selecting the exact winning order?

Solution

Here,

$$n = 20, \qquad r = 3.$$

Because the order of first, second, and third matters, we use permutations.

$$_{20}P_3 = \frac{20!}{(20-3)!} = \frac{20!}{17!}.$$

Cancel the common factor $17!$:

$$\frac{20!}{17!} = 20 \cdot 19 \cdot 18 = 6840.$$

So there are 6840 possible trifecta arrangements.
If one arrangement is randomly selected,

$$P(\text{correct winning order}) = \frac{1}{6840}.$$

In real horse racing, not all orders are equally likely because some horses tend to be faster than others.

6. Permutations with Identical Items

Sometimes some items are identical. In that case, the ordinary factorial rule overcounts the arrangements.

Permutations with Some Identical Items

If n items are available, and some are identical, then the number of different permutations is

$$\frac{n!}{n_1!n_2!\cdots n_k!},$$

where n_1 are alike, n_2 are alike, and so on.

Example 4: Survey Design: A survey has 10 questions. Two of the questions are identical to each other, and three other questions are also identical to each other. How many different arrangements of the survey questions are possible?

Solution

We have

$$n = 10.$$

Two questions are alike, so one repeated group has size 2. Three other questions are alike, so another repeated group has size 3.
Use

$$\frac{10!}{2!3!}.$$

Calculate:

$$\frac{10!}{2!3!} = \frac{3{,}628{,}800}{(2)(6)} = \frac{3{,}628{,}800}{12} = 302{,}400.$$

There are $302{,}400$ different arrangements.
This is far too many for a typical survey to use every possible arrangement.

7. Combinations Rule

Combinations Rule

When n different items are available and r of them are selected without replacement, the number of different combinations is

$$_nC_r = \frac{n!}{(n-r)!r!}.$$

Use this rule when order does not matter.

Example 5: Lottery: In Florida's Cash 5 lottery game, winning the jackpot requires selecting 5 different numbers from 1 to 35. The winning numbers can be drawn in any order.

(a) How many different lottery tickets are possible?

(b) Find the probability of winning the jackpot when one ticket is purchased.

Solution

Here,

$$n = 35, \qquad r = 5.$$

The order does not matter, so we use combinations.

$$_{35}C_5 = \frac{35!}{(35-5)!5!} = \frac{35!}{30!5!}.$$

Cancel $30!$:

$$_{35}C_5 = \frac{35 \cdot 34 \cdot 33 \cdot 32 \cdot 31}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}.$$

$$_{35}C_5 = 324{,}632.$$

So there are $324{,}632$ possible tickets.

If one ticket is purchased,

$$P(\text{winning jackpot}) = \frac{1}{324{,}632}.$$

8. Which Rule Should I Use?

Choosing a Counting Method

Situation	Method
Several stages, with choices at each stage	Multiplication counting rule
Arrange all n different items	Factorial rule: $n!$
Select r from n, order matters	Permutations: ${}_nP_r = \dfrac{n!}{(n-r)!}$
Select r from n, order does not matter	Combinations: ${}_nC_r = \dfrac{n!}{(n-r)!r!}$
Arrange items when some are identical	$\dfrac{n!}{n_1!n_2!\cdots n_k!}$

Practice: Decide whether each situation involves a permutation or a combination.

(a) Choosing 3 students from a class to serve on a committee.

(b) Choosing gold, silver, and bronze medal winners from 10 runners.

(c) Choosing 5 cards from a deck.

(d) Creating a 4-digit passcode with no repeated digits.

Solution

(a) Combination, because order does not matter for a committee.

(b) Permutation, because gold, silver, and bronze are different positions.

(c) Combination, because the same 5-card hand is the same regardless of order.

(d) Permutation, because position matters in a passcode.

Summary

Key Ideas

- The multiplication counting rule multiplies the number of choices at each stage.
- Factorials count arrangements of all different items.
- Permutations are used when order matters.
- Combinations are used when order does not matter.
- When some items are identical, divide by the factorials of the repeated groups.
- Counting methods help us find probabilities when outcomes are equally likely.