Chapter 5 Discrete Probability Distributions
Poisson Probability Distributions (5.3)

Objectives

In this lesson, we will learn how to

- recognize when a Poisson probability distribution is appropriate;
- use the Poisson probability formula;
- find and interpret Poisson probabilities;
- understand the mean and standard deviation of a Poisson distribution;
- use a Poisson distribution to approximate a binomial distribution when appropriate.

1 Key Idea

A Poisson probability distribution is a discrete probability distribution used for counting how many times an event occurs in a specified interval.

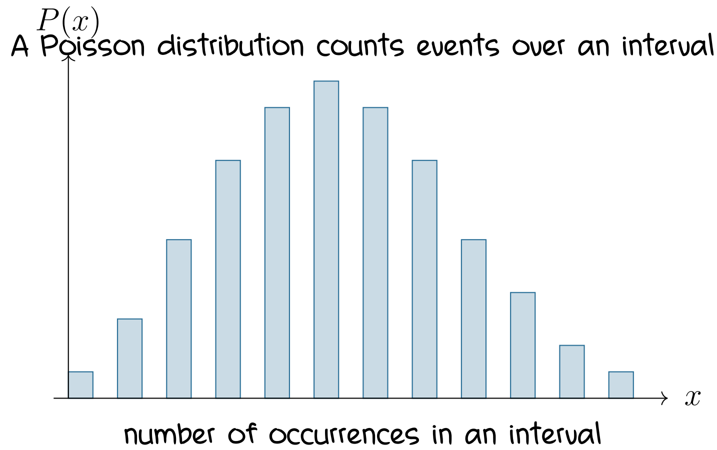
The interval can be a unit of time, distance, area, volume, or some similar unit.

Poisson Random Variable

The random variable x is the number of occurrences of an event in a specified interval.
Possible values are

$$x = 0, 1, 2, 3, \ldots$$

with no fixed upper limit.



2 Poisson Probability Formula

Poisson Probability Formula

If x is the number of occurrences of an event in an interval, then

$$P(x) = \frac{\mu^x e^{-\mu}}{x!},$$

where

- $e \approx 2.71828$,
- μ is the mean number of occurrences in the interval,
- $x = 0, 1, 2, 3, \ldots$.

3 Requirements for a Poisson Distribution

Requirements

A Poisson probability distribution is appropriate when:

1. The random variable x counts the number of occurrences of an event in some interval.
2. The occurrences are random.
3. The occurrences are independent of each other.
4. The occurrences are uniformly distributed over the interval.

4 Mean and Standard Deviation

A particular Poisson distribution is completely determined by the mean μ.

Parameters and Properties

For a Poisson distribution,

$$\text{mean} = \mu$$

and

$$\text{standard deviation} = \sigma = \sqrt{\mu}.$$

Example 1: There have been 652 Atlantic hurricanes during a 118-year period starting in 1900. Assume that the Poisson distribution is a suitable model.

(a) Find μ, the mean number of hurricanes per year.

(b) Find the probability that in a randomly selected year, there are exactly 6 hurricanes. That is, find $P(6)$.

(c) In this 118-year period, there were actually 16 years with 6 Atlantic hurricanes. How does this compare to the result in part (b)?

Solution

(a) The mean number of hurricanes per year is

$$\mu = \frac{652}{118} = 5.5254 \approx 5.5.$$

(b) Use the Poisson formula with $x = 6$ and $\mu = 5.5$:

$$P(6) = \frac{5.5^6 e^{-5.5}}{6!}.$$

Using technology or a calculator,

$$P(6) \approx 0.157.$$

So the probability of exactly 6 hurricanes in a year is about

$$0.157.$$

(c) In 118 years, the expected number of years with exactly 6 hurricanes is

$$118(0.157) = 18.526 \approx 18.5.$$

The actual number was 16 years, which is reasonably close to 18.5. Therefore, the Poisson model appears to work quite well in this case.

5 Poisson Approximation to the Binomial Distribution

Sometimes a Poisson distribution can be used to approximate a binomial distribution.

This is useful when the number of trials is large, but the probability of success on one trial is small.

Poisson Approximation to Binomial

We can use the Poisson distribution to approximate the binomial distribution when both conditions are satisfied:

$$n \geq 100$$

and

$$np \leq 10.$$

When using this approximation, use

$$\mu = np.$$

Example 2: In the Maine Pick 4 game, you pay 50 cents to select a sequence of four digits, such as 1377. If you play this game once every day for 365 days, find the probability of winning at least once in a year.

Solution

There are $10{,}000$ possible four-digit sequences from 0000 to 9999. The probability of winning on one play is

$$p = \frac{1}{10{,}000} = 0.0001.$$

Since the game is played once per day for 365 days,

$$n = 365.$$

Check the requirements for the Poisson approximation:

$$n = 365 \geq 100$$

and

$$np = 365(0.0001) = 0.0365 \leq 10.$$

So we may use the Poisson approximation with

$$\mu = np = 0.0365.$$

We want the probability of winning at least once. It is easier to use the complement:

$$P(\text{at least one win}) = 1 - P(\text{no wins}).$$

First find $P(0)$:

$$P(0) = \frac{(0.0365)^0 e^{-0.0365}}{0!}.$$

Since $(0.0365)^0 = 1$ and $0! = 1$,

$$P(0) = e^{-0.0365} \approx 0.9642.$$

Therefore,

$$P(\text{at least one win}) = 1 - 0.9642 = 0.0358.$$

So the probability of winning at least once in a year is about

$$0.0358,$$

or about 3.58%.

6 Common Strategy: At Least One

At Least One Strategy

When a problem asks for the probability of at least one occurrence, use the complement:

$$P(\text{at least one}) = 1 - P(\text{none}).$$

This strategy is often much easier than adding many probabilities.

Practice: Suppose a Poisson distribution has mean $\mu = 2.4$. Find the probability that there is at least one occurrence in the interval.

Solution

Use the complement:

$$P(\text{at least one}) = 1 - P(0).$$

For $\mu = 2.4$,

$$P(0) = \frac{2.4^0 e^{-2.4}}{0!} = e^{-2.4} \approx 0.0907.$$

Therefore,

$$P(\text{at least one}) = 1 - 0.0907 = 0.9093.$$

Summary

Key Ideas

- A Poisson distribution counts occurrences of an event in an interval.
- The Poisson formula is

$$P(x) = \frac{\mu^x e^{-\mu}}{x!}.$$

- For a Poisson distribution, the mean is μ, and the standard deviation is $\sqrt{\mu}$.
- A Poisson model requires random, independent, uniformly distributed occurrences.
- A Poisson distribution can approximate a binomial distribution when $n \geq 100$ and $np \leq 10$, using $\mu = np$.
- For ``at least one,'' it is usually easiest to use

$$P(\text{at least one}) = 1 - P(\text{none}).$$