Chapter 6 Normal Probability Distributions
The Standard Normal Distribution (6.1)

1 Objectives

In this lesson, we will learn how to

- recognize normal, uniform, and density curves;
- understand the standard normal distribution;
- find probabilities from given z scores;
- find probabilities to the left, to the right, and between two z scores;
- find z scores when areas are given;
- understand and use critical values such as z_α.

2 Continuous Probability Distributions

A continuous random variable can take any value in an interval. For example, a waiting time could be

$$1.2 \text{ min}, \quad 1.23 \text{ min}, \quad 1.234 \text{ min}, \quad \text{and so on.}$$

For continuous probability distributions, probabilities are represented by areas under a curve.

Important Idea

For a continuous probability distribution,

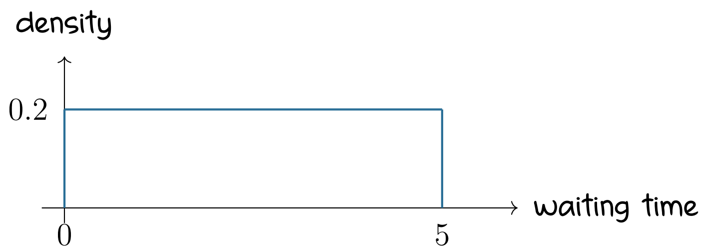
$$\text{Probability} = \text{Area under the curve}.$$

The total area under the curve must be exactly 1.

3 Uniform Distribution

A continuous random variable has a uniform distribution if all values in an interval are equally likely.

The graph of a uniform distribution is rectangular.



Height of a Uniform Distribution

If a uniform distribution covers an interval of length called the range, then

$$\text{height} = \frac{1}{\text{range}}.$$

This makes the total area equal to 1.

Example 1: During certain time periods, airport security waiting times are uniformly distributed from 0 minutes to 5 minutes. Find the probability that a randomly selected passenger waits at least 2 minutes.

Solution

The range is

$$5 - 0 = 5.$$

So the height is

$$\frac{1}{5} = 0.2.$$

``At least 2 minutes'' means

$$2 \le x \le 5.$$

The width of this interval is

$$5 - 2 = 3.$$

Therefore,

$$P(x \geq 2) = \text{area} = \text{height} \times \text{width}$$

$$= 0.2(3) = 0.6.$$

The probability is

$$\boxed{0.6}.$$

4 Density Curve

The graph of any continuous probability distribution is called a density curve.

Density Curve

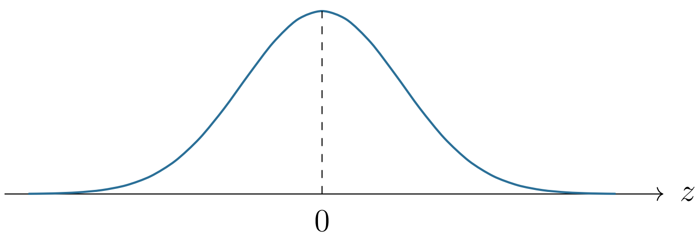
A density curve must satisfy two conditions:

- The curve is on or above the horizontal axis.
- The total area under the curve is exactly 1.

Because the total area is 1, we can interpret areas as probabilities.

5 Normal Distribution

A continuous random variable has a normal distribution if its graph is symmetric and bell-shaped.



A normal curve is symmetric and bell-shaped.

Normal Distribution

A normal distribution is described by two parameters:

- Mean: μ
- Standard deviation: σ

The mean controls the center of the curve, and the standard deviation controls the spread.

6 Standard Normal Distribution

The standard normal distribution is a special normal distribution with

$$\mu = 0 \qquad \text{and} \qquad \sigma = 1.$$

Standard Normal Distribution

The standard normal distribution has three key properties:

- It is bell-shaped.
- It has mean $\mu = 0$.
- It has standard deviation $\sigma = 1$.

The total area under the curve is 1.

A value on the horizontal axis of the standard normal distribution is called a z score.

z Score

A z score tells how many standard deviations a value is above or below the mean.

- A positive z score is above the mean.
- A negative z score is below the mean.
- $z = 0$ is exactly at the mean.

7 Finding Probabilities from Given z Scores

To find a probability from a z score, we usually use technology or a standard normal table.

Using Table A-2

Table A-2 gives cumulative area from the left. That means the table gives

$$P(z < a).$$

So if we need an area to the right or between two values, we must adjust the table value.

Do Not Confuse These

- A z score is a location on the horizontal axis.
- An area is a region under the curve.
- An area represents probability.

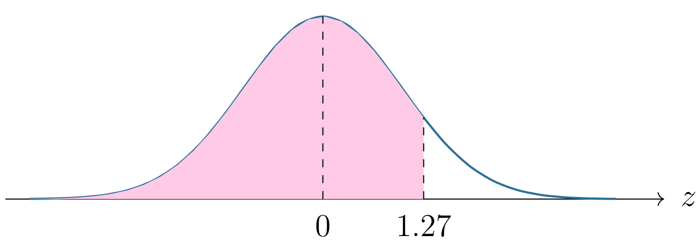
8 Area to the Left

Example 2: Bone density test scores are normally distributed as a standard normal distribution. Find the probability that a randomly selected adult has a bone density test score less than 1.27.

Solution

We want

$$P(z < 1.27).$$



Area to the left of $z = 1.27$

Using the standard normal table or technology,

$$P(z < 1.27) = 0.8980.$$

Therefore,

$$\boxed{P(z < 1.27) = 0.8980}.$$

Interpretation:
About

$$89.80\%$$

of values are below $z = 1.27$.

9 Area to the Right

If we need an area to the right, remember that the total area under the curve is 1.

Area to the Right

$$P(z > a) = 1 - P(z < a).$$

Example 3: Using the same bone density test scores, find the probability that a randomly selected person has a result above -1.00.

Solution

We want

$$P(z > -1.00).$$

Table A-2 gives the area to the left:

$$P(z < -1.00) = 0.1587.$$

Therefore,

$$P(z > -1.00) = 1 - 0.1587.$$

$$P(z > -1.00) = 0.8413.$$

So,

$$\boxed{P(z > -1.00) = 0.8413}.$$

Interpretation:
About

$$84.13\%$$

of values are above $z = -1.00$.

10 Area Between Two Values

To find the area between two z scores, subtract the two cumulative areas from the left.

Area Between Two z Scores

If $a < b$, then

$$P(a < z < b) = P(z < b) - P(z < a).$$

Example 4: A bone density reading between -2.50 and -1.00 indicates osteopenia, which means some bone loss. Find the probability that a randomly selected subject has a reading between -2.50 and -1.00.

Solution

We want

$$P(-2.50 < z < -1.00).$$

Find the cumulative areas:

$$P(z < -1.00) = 0.1587,$$

$$P(z < -2.50) = 0.0062.$$

Subtract:

$$P(-2.50 < z < -1.00) = 0.1587 - 0.0062.$$

$$= 0.1525.$$

Therefore,

$$\boxed{P(-2.50 < z < -1.00) = 0.1525}.$$

Interpretation:
About

$$15.25\%$$

of values are between $z = -2.50$ and $z = -1.00$.

11 Probability Notation

Common Probability Notation

$P(z < a)$ means the probability that z is less than a.

$P(z > a)$ means the probability that z is greater than a.

$P(a < z < b)$ means the probability that z is between a and b.

12 Finding z Scores from Known Areas

Sometimes we are given an area and asked to find the corresponding z score.

Finding a z Score from an Area

1. Draw a bell-shaped curve.
2. Shade the region that matches the given probability.
3. Convert the region to a cumulative area from the left if needed.
4. Use technology or Table A-2 to find the corresponding z score.

Example 5: Find the z score with cumulative area 0.9750 to its left.

Solution

We want the value of z such that

$$P(Z < z) = 0.9750.$$

Using Table A-2 or technology,

$$z = 1.96.$$

Therefore,

$$\boxed{z = 1.96}.$$

13 Critical Values

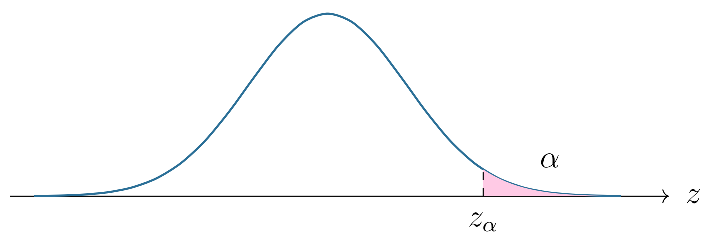
A critical value is a z score that separates common values from significantly low or significantly high values.

Critical Value Notation

The notation

$$z_\alpha$$

means the z score with area α to its right.



Example 6: Find the value of $z_{0.025}$.

Solution

The notation $z_{0.025}$ means the z score with area

$$0.025$$

to its right.
Since Table A-2 gives area to the left, convert the right-tail area to a left-tail area:

$$1 - 0.025 = 0.975.$$

Now find the z score with cumulative area 0.975 to the left.

$$z = 1.96.$$

Therefore,

$$\boxed{z_{0.025} = 1.96}.$$

Caution

For z_α, the area α is to the right of the z score. However, Table A-2 gives cumulative area to the left.

So use

$$1 - \alpha$$

when looking up z_α in the table.

Summary

Key Ideas

- A density curve has total area 1.
- For continuous distributions, probability is area under the curve.
- A normal distribution is symmetric and bell-shaped.
- The standard normal distribution has $\mu = 0$ and $\sigma = 1$.
- Table A-2 gives cumulative area from the left.
- To find an area to the right, subtract the left area from 1.
- To find an area between two values, subtract two cumulative left areas.
- z_α means the z score with area α to its right.