Module 8 Lesson 2
Testing a Claim About a Proportion (8.2)

Objectives

In this lesson, we will learn how to

- test a claim about a population proportion using the normal approximation method;
- identify the number of successes from a sample proportion;
- use the P-value method and the critical value method;
- understand why confidence intervals are not always equivalent to hypothesis tests for proportions;
- use exact binomial methods for testing claims about proportions.

1. Key Idea

This section gives a complete procedure for testing a claim about a population proportion p.

The same methods can be used for claims about proportions, probabilities, or decimal equivalents of percentages.

Main Goal

We use sample information to decide whether there is enough evidence to support or reject a claim about a population proportion p.

2. Finding the Number of Successes

When technology is used, we usually enter two values:

$$n = \text{sample size}, \qquad x = \text{number of successes}.$$

Sometimes a problem gives the sample proportion $\hat{p}$ instead of x. In that case, use

$$x = n\hat{p}.$$

Because x counts people or objects, it must be rounded to a whole number.

Example: If 52% of 926 survey respondents answer ``yes,'' then

$$x = 926(0.52) = 481.52 \approx 482.$$

So we use $x = 482$ successes.

Caution

Slightly different results may occur if we use a rounded sample proportion such as 0.52 instead of the exact value

$$\hat{p} = \frac{482}{926}.$$

Technology usually uses the whole-number count x.

3. Confidence Intervals Are Not Always Equivalent

For tests about proportions, the confidence interval method is not always equivalent to the P-value method or the critical value method.

Important Recommendation

Use confidence intervals to estimate a population proportion.
Use the P-value method or critical value method to test a claim about a population proportion.

Why? The P-value and critical value methods use the claimed value of p from H_0. A confidence interval uses $\hat{p}$ from the sample.

4. Testing a Claim About a Proportion

Requirements

To test a claim about a population proportion p using the normal approximation method:

1. The sample observations form a simple random sample.
2. The conditions for a binomial distribution are satisfied:
   - fixed number of trials;
   - independent trials;
   - two categories: success and failure;
   - constant probability of success.
3. The conditions

$$np \geq 5 \qquad \text{and} \qquad nq \geq 5$$

   are both satisfied, where p is the value assumed in the null hypothesis and $q = 1 - p$.

5. Test Statistic

Test Statistic for a Proportion

For a hypothesis test about a population proportion,

$$z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}},$$

where

$$\hat{p} = \frac{x}{n}, \qquad q = 1 - p.$$

Here p is the claimed population proportion from the null hypothesis.

P-values can be found by technology or by using the standard normal distribution table.

6. Example: Two-Factor Authentication

Example 1: In a survey of 926 Internet users, 52% of them responded ``yes'' when asked whether they use two-factor authentication.
Use these results to test the claim that most Internet users use two-factor authentication to protect their online data. Interpret ``most'' as more than half.
Use a significance level of $\alpha = 0.05$.

Solution

The claim ``most Internet users use two-factor authentication'' means

$$p > 0.5.$$

Because 52% of 926 respondents said yes,

$$x = 926(0.52) = 481.52 \approx 482.$$

Thus,

$$\hat{p} = \frac{482}{926} = 0.5205.$$

Requirement check:

- The respondents are treated as randomly selected.
- There is a fixed number of trials, $n = 926$.
- Each response has two categories: yes or no.
- Under the null hypothesis, $p = 0.5$ and $q = 0.5$.

Check the normal approximation requirements:

$$np = 926(0.5) = 463 \geq 5,$$

$$nq = 926(0.5) = 463 \geq 5.$$

The requirements are satisfied.

P-Value Method

Solution

Step 1: The original claim is

$$p > 0.5.$$

Step 2: The opposite claim is

$$p \leq 0.5.$$

Step 3: The hypotheses are

$$H_0 : p = 0.5,$$

$$H_1 : p > 0.5.$$

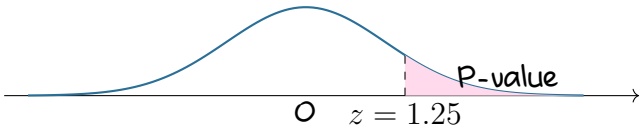
This is a right-tailed test.

Step 4: Use

$$\alpha = 0.05.$$

Step 5: Since this is a test about a proportion, use the normal distribution.
Step 6: The test statistic is

$$z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}} = \frac{0.5205 - 0.5}{\sqrt{\frac{(0.5)(0.5)}{926}}} \approx 1.25.$$



Solution

For this right-tailed test, the P-value is the area to the right of $z = 1.25$. Using technology,

$$\text{P-value} \approx 0.1059.$$

Using Table A-2 with rounded $z = 1.25$ gives

$$1 - 0.8944 = 0.1056.$$

Step 7: Since

$$\text{P-value} = 0.1059 > 0.05,$$

we fail to reject H_0.
Step 8: There is not sufficient sample evidence to support the claim that most Internet users use two-factor authentication to protect their online data.

Critical Value Method

Critical Value Method

For a right-tailed test with $\alpha = 0.05$, the critical value is

$$z = 1.645.$$

Reject H_0 only if the test statistic is in the critical region.

Solution

The test statistic is

$$z = 1.25.$$

The critical value is

$$z = 1.645.$$

Because

$$1.25 < 1.645,$$

the test statistic is not in the critical region.
Therefore, we fail to reject H_0.
Final conclusion: There is not sufficient evidence to support the claim that most Internet users use two-factor authentication.

Confidence Interval Method

A Note About Confidence Intervals

For a right-tailed proportion test with $\alpha = 0.05$, the related confidence interval would use a 90% confidence level.

Solution

Using $n = 926$ and $x = 482$, a 90% confidence interval is approximately

$$0.494 < p < 0.548.$$

Because this interval includes values below 0.5 and also includes values

above 0.5, we cannot safely conclude that

$$p > 0.5.$$

This agrees with the P-value and critical value methods in this example.

7. Example: Fewer Than 30% of Adults Have Sleepwalked?

Example 2: A study reported that 29.2% of $19{,}136$ American adults have sleepwalked.
Use a significance level of $\alpha = 0.05$ to test the claim that the population proportion of adults who have sleepwalked is less than 0.30.

Solution

Requirement check:

- The sample is treated as a simple random sample.
- There is a fixed number of trials: $n = 19{,}136$.
- Each subject either has sleepwalked or has not sleepwalked.
- Using the claimed value $p = 0.30$, we have $q = 0.70$.

Check:

$$np = 19{,}136(0.30) = 5{,}740.8 \geq 5,$$
$$nq = 19{,}136(0.70) = 13{,}395.2 \geq 5.$$

The requirements are satisfied.

Solution

Step 1: The original claim is

$$p < 0.30.$$

Step 2: The opposite claim is

$$p \geq 0.30.$$

Step 3: The hypotheses are

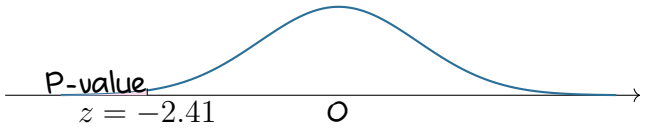
$$H_0 : p = 0.30,$$

$$H_1 : p < 0.30.$$

This is a left-tailed test.
Step 4: Use

$$\alpha = 0.05.$$

Step 5: Since this is a test about a population proportion, use the normal approximation.



Solution

Step 6: The sample proportion is

$$\hat{p} = 0.292.$$

The test statistic is

$$z = \frac{0.292 - 0.30}{\sqrt{\frac{(0.30)(0.70)}{19136}}} \approx -2.41.$$

For this left-tailed test, the P-value is the area to the left of $z = -2.41$:

$$\text{P-value} = 0.0080.$$

Step 7: Since

$$0.0080 \leq 0.05,$$

we reject H_0.
Step 8: There is sufficient evidence to support the claim that fewer than 30% of adults have sleepwalked.

8. Exact Methods for Proportions

The normal approximation method is useful for introducing the main ideas of hypothesis testing. However, exact methods use the binomial distribution directly.

Exact Binomial Method

Identify:

$n =$ sample size, $x =$ number of successes, $p =$ claimed proportion.

Then use binomial probabilities:

- Left-tailed test:

$$\text{P-value} = P(x \text{ or fewer successes}).$$

- Right-tailed test:

$$\text{P-value} = P(x \text{ or more successes}).$$

- Two-tailed test:

$$\text{P-value} = 2(\text{smaller tail probability}).$$

Example 3: In the sleepwalking example, 29.2% of $19{,}136$ adults have sleepwalked. This gives about

$$x = 5588$$

successes. Use the exact binomial method to test the claim

$$p < 0.30.$$

Solution

Because this is a left-tailed test, use

$$\text{P-value} = P(5588 \text{ or fewer successes among } 19136 \text{ trials}),$$

assuming

$$p = 0.30.$$

Technology gives

$$\text{P-value} = 0.0080286.$$

Since

$$0.0080286 < 0.05,$$

we reject H_0.
Conclusion: There is sufficient evidence to support the claim that fewer than 30% of adults have sleepwalked.

9. Continuity Correction for the Exact Method

Improving the Exact Method

The exact method can be conservative, meaning the true probability of a Type I error may be less than the selected significance level α.
A simple continuity correction adjusts the P-value by subtracting one-half of the binomial probability at the boundary.

For introductory statistics, the most important idea is to understand when to use:

- the normal approximation method;
- the exact binomial method;
- technology for binomial probabilities.

Summary

Key Ideas

- For a test about a population proportion, the null hypothesis uses an equality statement, such as $H_0 : p = 0.5$.
- The test statistic is
$$z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}}.$$
- Use the claimed value of p from H_0 when checking $np \geq 5$ and $nq \geq 5$.
- The P-value method compares the P-value with α.
- The critical value method compares the test statistic with the critical value.
- For tests about proportions, confidence intervals are not always equivalent to P-value or critical value tests.
- Exact methods use binomial probabilities directly and are usually found using technology.