Module 8 Lesson 3
Testing a Claim About a Mean (8.3)

Objectives

In this lesson, we will learn how to

- test a claim about a population mean μ when the population standard deviation σ is not known;
- decide when to use the Student t distribution;
- identify the correct hypotheses, test statistic, P-value, and conclusion;
- compare the P-value method, critical value method, and confidence interval method;
- briefly recognize the less common case when σ is known.

1. Key Idea

Testing a claim about a population mean is one of the most common and important hypothesis-testing methods.

In realistic applications, the population standard deviation σ is usually not known. In that case, we use the sample standard deviation s and the Student t distribution.

Main Situation in This Section

We test claims about a population mean μ when σ is not known.

2. Notation

Notation for Testing a Claim About μ

n	sample size
$\bar{x}$	sample mean
s	sample standard deviation
μ	population mean

3. Requirements

Requirements for Testing a Claim About μ when σ is Unknown

1. The sample is a simple random sample.
2. Either the population is normally distributed or $n > 30$.

If $n \leq 30$, we should check whether the sample data appear to come from a normally distributed population. A histogram or normal quantile plot can help.

4. Test Statistic

When σ is not known, use the sample standard deviation s and the Student t distribution.

Test Statistic for Testing a Claim About a Mean

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

Use degrees of freedom

$$df = n - 1.$$

P-values and Critical Values

- P-values may be found using technology or the Student t distribution table.
- Critical values are found using the Student t distribution with $df = n - 1$.

5. The Student t Distribution

Important Properties of the Student t Distribution

- It is different for different sample sizes.
- It has the same general bell shape as the standard normal distribution.
- It is wider than the standard normal distribution because s is used to estimate σ.
- It has mean $t = 0$.
- Its standard deviation is greater than 1.
- As n increases, the Student t distribution gets closer to the standard normal distribution.

6. Example: Adult Sleep

Example 1: The following sleep times, in hours, are from 12 randomly selected adults:

$$4,\ 8,\ 4,\ 4,\ 8,\ 6,\ 9,\ 7,\ 7,\ 10,\ 7,\ 8.$$

For this sample,

$$n = 12, \qquad \bar{x} = 6.83333333, \qquad s = 1.99240984.$$

A common recommendation is that adults should sleep at least 7 hours. Use the P-value method with significance level $\alpha = 0.05$ to test the claim that the mean amount of adult sleep is less than 7 hours.

Solution

Requirement check. The sample is treated as a simple random sample. Since $n = 12$ does not exceed 30, we should check whether the data appear to come from a normally distributed population. Based on a histogram/normal quantile plot and no obvious outliers, this requirement is reasonable.

Step 1. State the claim.

$$\mu < 7.$$

Step 2. Write the opposite of the claim.

$$\mu \geq 7.$$

Step 3. Write the hypotheses.

$$H_0 : \mu = 7$$

$$H_1 : \mu < 7.$$

This is a left-tailed test.
Step 4. Significance level.

$$\alpha = 0.05.$$

Step 5. Test statistic. Because σ is unknown, use the Student t distribution with

$$df = n - 1 = 12 - 1 = 11.$$

Step 6. Compute the test statistic.

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{6.83333333 - 7}{1.99240984/\sqrt{12}} \approx -0.290.$$

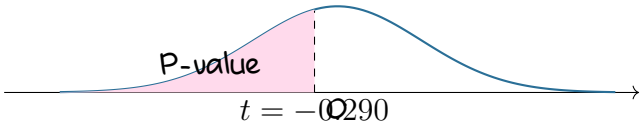
Technology gives

$$\text{P-value} \approx 0.3887.$$

Step 7. Decision. Since

$$0.3887 > 0.05,$$

we fail to reject H_0.
Step 8. Conclusion. There is not sufficient evidence to support the claim that the mean amount of adult sleep is less than 7 hours.



7. P-Value Method Without Technology

If technology is not available, Table A-3 can be used to estimate a range for the P-value.

For the previous adult sleep example:

$$t = -0.290, \qquad df = 11.$$

Because Table A-3 is designed for positive t values and right-tail areas, we use the positive value 0.290 and interpret it for the left tail.

Using Table A-3

For $df = 11$, the value 0.290 is smaller than the listed values in the row, so the left-tail P-value is greater than 0.10.
This is enough to conclude that the P-value is greater than $\alpha = 0.05$, so we fail to reject H_0.

8. Critical Value Method

For the adult sleep example, the test is left-tailed with

$$\alpha = 0.05, \qquad df = 11.$$

From Table A-3, the critical value for an area of 0.05 in one tail is

$$t = 1.796.$$

Since the test is left-tailed, the actual critical value is

$$t = -1.796.$$

Decision by Critical Value Method

The test statistic is

$$t = -0.290.$$

It does not fall in the critical region, because

$$-0.290 > -1.796.$$

Therefore, we fail to reject H_0.

9. Confidence Interval Method

For a left-tailed test with significance level $\alpha = 0.05$, the comparable confidence interval uses a confidence level of

$$1 - 2\alpha = 1 - 2(0.05) = 0.90.$$

For the adult sleep data, the 90% confidence interval is

$$5.80 < \mu < 7.87.$$

Interpretation

Since the null value $\mu = 7$ is inside the confidence interval, 7 hours is still a plausible value for the population mean. Therefore, we fail to reject H_0.

10. Example: Body Temperature

Example 2: A data set of body temperatures has the following summary statistics:

$$n = 106, \qquad \bar{x} = 98.20°\text{F}, \qquad s = 0.62°\text{F}.$$

Use significance level $\alpha = 0.05$ to test the common belief that the population mean body temperature is $98.6°$F.

Solution

Requirement check. The sample can be treated as a simple random sample. Since $n = 106 > 30$, the normality/sample-size requirement is satisfied.

Step 1. State the claim.

$$\mu = 98.6.$$

Step 2. Write the opposite of the claim.

$$\mu \neq 98.6.$$

Step 3. Write the hypotheses.

$$H_0 : \mu = 98.6$$

$$H_1 : \mu \neq 98.6.$$

This is a two-tailed test.

Step 4. Significance level.

$$\alpha = 0.05.$$

Step 5. Test statistic. Since σ is unknown, use the Student t distribution with

$$df = n - 1 = 105.$$

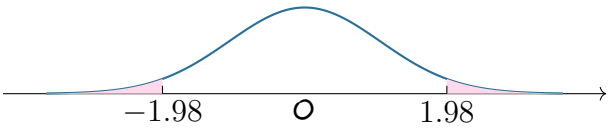
Step 6. Compute the test statistic.

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{98.20 - 98.60}{0.62/\sqrt{106}} \approx -6.64.$$

Technology gives a P-value essentially equal to 0.

Step 7. Decision. Since the P-value is less than 0.05, reject H_0.

Step 8. Conclusion. There is sufficient evidence to reject the common belief that the population mean body temperature is $98.6°$F.



11. Three Equivalent Methods for Means

For testing a claim about a mean, the following methods lead to the same conclusion when used correctly:

Methods for Testing a Claim About μ

- P-value method
- Critical value method
- Confidence interval method

For the body temperature example:

- P-value method: P-value is very small, so reject H_0.
- Critical value method: $t = -6.64$ falls in the critical region, so reject H_0.
- Confidence interval method: the 95% confidence interval is

$$98.08°\text{F} < \mu < 98.32°\text{F}.$$

Since $98.6°$F is not in this interval, reject H_0.

12. When σ Is Known

The case where the population standard deviation σ is known is rare. If σ is known, use the standard normal distribution instead of the Student t distribution.

Requirements When σ Is Known

1. The sample is a simple random sample.
2. The population standard deviation σ is known.
3. Either $n > 30$ or the population is normally distributed.

Test Statistic When σ Is Known

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

Use the standard normal distribution.

Summary

Key Ideas

- When testing a claim about μ and σ is unknown, use the Student t distribution.
- The test statistic is
$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}.$$
- Use degrees of freedom $df = n - 1$.
- Check that the sample is simple random and that the population is normal or $n > 30$.
- A left-tailed test uses $H_1 : \mu <$ claimed value.
- A right-tailed test uses $H_1 : \mu >$ claimed value.
- A two-tailed test uses $H_1 : \mu \neq$ claimed value.
- For means, the P-value, critical value, and confidence interval methods should agree.
- If σ is known, use a z test instead of a t test.